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COUPLING COEFFICIENTS AND TENSOR OPERATORS FOR CHAINS OF GROUPS

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CONTENTS

The action of an arbitrary (but finite or compact) group on an arbitrary Hilbert space is studied. The application of group theory to physical calculations is often based on the Wigner–Eckart theorem, and one of the aims is to lead up to a general proof of this theorem.

The group's action gives irreducible ket-vector representation spaces, products of which lead to a definition of coupling (Wigner, or Clebsch–Gordan) coefficients and *jm* and *j* symbols. The properties of these objects are studied in detail, beginning with properties that are independent of the basis chosen for the representation spaces. We then explore some of the consequences of choosing bases by using the action of a subgroup. This leads to the Racah factorization lemma and the definition of *jm* factors, also a general statement of Racah's reciprocity.

In the third part, we add to these ideas, some properties of the space of all linear operators taking the Hilbert space to itself. This leads to a proof of the Wigner–Eckart theorem which is both succinct and in the language of quantum mechanics.

1. Introduction

In all the many applications of group theory to physics, one starts with two concepts, a Hilbert space and a group of operators acting on that space. The Hilbert space is the space of wave functions of the system, or some subspace theoreof. The group, however, is chosen in many different ways. In crystal theory, for instance, it is usual to take as the group, those space operations on the atom which leave it invariant (Hamermesh 1962, p. xiii). The choice of group which Racah made (1942*a*, *b*, 1943, 1949) stirred great interest among many branches of theoretical physics. In studying the atomic f-shell, Racah separated the Hamiltonian into a basic Hamiltonian, H_0, and a perturbation H_1. The H_0 factorizes into radial and angular parts, which may be solved separately, the angular part giving rise to spherical harmonics. This much was standard (Condon & Shortley 1935) but previous attempts at introducing the perturbation H_1 had been foiled by the multitude of angular functions (3432 for the half-filled f-shell). Racah showed how to proceed: after writing the Hamiltonian in terms of 'effective' operators acting only on the angular functions, he first noted that the (first order) operators generated the group U_{14} and that the subgroup chain

$$U_{14} \supset Sp_{14} \supset SU_2 \times (R_7 \supset G_2 \supset R_3 \supset R_2)$$

was a chain of approximate symmetries (this fact aids the computation but is not essential). He then used group theory to obtain the matrix elements of the Hamiltonian $H_0 + H_1$, in terms of an eigen-basis for H_0. Judd (1963) describes the mathematics involved, including many later developments. Wybourne (1965) was subsequently able to use these ideas to systematize the spectral properties of the rare-earth atoms and ions.

The advent of computers which enabled a straightforward application of the Condon & Shortley (1935) techniques, was the reason why Racah's ideas were not universally accepted by atomic physicists. However Racah's 1949 paper had profound consequences in nuclear theory with Jahn (1950) applying the techniques immediately. Nuclear physicists in their search for a satisfactory many particle theory have explored many approaches: some work with effective operators – operators which only have an action defined on certain wavefunctions, and others choose the operators to be those of position and/or momentum. The distinction can become rather hazy, such as regards operators on spin-space. Moshinsky's recent (1973) suggestion of working in 'phase-space' is perhaps the most fascinating. His idea to take as coordinates for each of the *n* particles both the position and the momentum gives rise to a symplectic group, Sp_{6n}.

In every application one wants more than just the representation structure of the group, one wants the matrix elements of a large number of operators. In Racah's treatment one requires these for operators (on the Hilbert space) which are not in the group. Judd (1966) used the quasi-particle formulation to expand the group from U_{14} to $U_{2^{14}}$ ensuring that all operators of interest were in the group. In the third part we study operators, choose various bases and prove the Wigner–Eckart theorem for an arbitrary group. The Wigner–Eckart theorem gives the matrix elements of the tensor basis operators between basis kets, as a coupling coefficient (or 3-*jm* symbol). First we must focus on the properties of the coupling coefficients.

Those physicists who work alongside pure mathematicians are aware that there are these two distinct questions one may ask

(1) What general structures exist?

(2) What are the numerical values of the various coefficients and matrix elements which occur within these structures?

We note we require an answer to the second question.

Mathematicians have focused their attention on the first question. The names of E. Cartan, I. Schur, A. Young and H. Weyl stand out among the originators. Their techniques have been developed over the years and the subject is almost closed in so far as one is concerned with representations of compact or finite groups over a field of characteristic zero. In § 7 we shall be giving some recent references to this subject of character theory. These references will be to computation techniques, as distinct from structural theorems.

Few physicists have looked at the second question in general, although there have been many attempts to study certain classes of groups. Such restricted studies tend to obscure many essential properties by making general results appear to depend on the group chosen. As the answers to the most basic questions (such as – how much freedom is there in the phase choice of a 3-*jm* symbol for a given group?) are either unknown or not well known, the simplest many-particle calculations run into relatively abstract group theoretic problems. This has perhaps been the prime cause of the circumstance of which Ne'eman writes (1964): 'The reputation that symmetries somehow acquired – of being able to predict only qualitative results – is libellous.'

It is our purpose, in this article, to attempt to redress this situation. Our approach will be to consider question two, taking the answer to question one as known. Both for reasons of mathematical simplicity and because of their present greater use, we shall restrict our attention to finite, or compact Lie, groups. In this case, not only is the answer to question one almost completely known, but also all irreducible unitary representations are finite dimensional. (Finite dimensionality of the Hilbert space is usually used in perturbation theories.) Thus, although non-compact groups have certainly been shown to be useful (for some references see Wulfman 1971) by an arbitrary group we shall always mean a finite or a compact group.

Using this restriction of the types of group we shall consider, and by studying the action of a group on a (separable) Hilbert space, we shall be able to explore the properties in a rather simple manner, neither using much abstract group theory, nor using properties of vector spaces that are not well known to every physicist and chemist.

One of the unfortunate consequences of the variety of appliactions is that in the literature there is a multitude of names and notations – Clebsch–Gordan, Wigner, vector-coupling, fractional parentage, isoscalar factors, etc., ..., for essentially the same matrix elements. Now A. Clebsch and P. Gordan only studied question one, and our subject can be dated from Wigner's privately circulated manuscript of 1940 (Wigner 1940). Wigner gave a general discussion of the coefficients for simply reducible groups. His results have been extended in various directions over the years. Notable work has appeared in the books by Hamermesh (1962, for the symmetric group), by Griffith (1962, for crystallographic groups) and by Vanagas (1971, for most groups). To these must be added the paper by Racah (1949) and many papers by Baird, Biedenharn, Louck, Moshinsky and their co-workers on the unitary groups (see Louck (1970) and Moshinsky & Devi (1969) for detailed references). Harnung & Schäffer (1973*a*, *b*) and Harnung (1973) have continued Griffith's work, and looked at questions of reality. However, only the papers by Derome & Sharp (Derome & Sharp 1965; Derome 1966) treat the problem for an arbitrary group. It is their general analysis we wish to extend.

We have another remark on notation. In the theory of angular momentum one refers to the symmetrized coupling and recoupling coefficients as 3-*j* and 6-*j* symbols respectively. This is clearly bad notation, for the 3-*j* symbol depends on the basis labels (the *m* quantum number) whereas the 6-*j* and higher symbols do not. We shall follow recent usage in calling the 1-*j* and 3-*j* symbols *jm* symbols.

This article is divided into three parts. The first and much the larger part is devoted to systematic definitions and derivation of the properties of ket representations, coupling (Wigner) and recoupling (Racah) coefficients, 1-*jm* and 3-*jm* symbols, the 1-*j* phases, the 3-*j* permutation matrices and the 6-*j* and 9-*j* symbols. We note the freedom we have in their definitions, and use this freedom to simplify their properties. The results of Derome (1966) on the symmetries of the coefficients are used and extended, thus removing a few of the arbitrary choices of phase. Section 7 uses character theory to illustrate these results for many groups. We are thus able to show that there is little difference in these properties between quite different groups, excepting groups of small dimension when one or two simplifications occur. In § 8, we elaborate on Derome & Sharp's (1965) results on the raising and lowering of the indices of the 3-*jm* symbol. In a paper on character theory elsewhere (Butler & King 1974) we prove a result on 1-*j* phases (equation (8.10)) for all Lie groups and for most finite groups. This particular result allows us to simplify the results of Derome & Sharp for all but these rare cases, because the position of the multiplicity index can then be shown to be unimportant.

In the first part we use no information concerning the bases of the representations, but in the second we explore the factorization property which always follows when a subgroup is imbedded in the group – an operation which (partially) fixes the basis. This part is relatively brief, because the results follow almost trivially from the former, and certainly not because the results are unimportant. Indeed the factorization lemma of Racah (1949) gives many simplifications in physical applications, as well as being the basis of most methods for actually computing coefficients. Thus the equation (13.13) giving the symmetries of the isoscalar factors may be regarded as the most important single result. However, this factorizability has largely been ignored in previous analytic work, perhaps because it requires the introduction of additional symbols in equations which are already bedevilled by the number of symbols.

The subgroup structure simplifies the problems of computing the coupling coefficients, to the extent that if we know the 3-*jm* factors for each step in a canonical chain, we may trivially write down any entire coefficient. In physical applications, when canonical chains are not often used, it is possible to transform to another basis as the last step. (See, for example, Kaplan 1962*a*, *b*; Moshinsky & Devi 1969; Kramer 1968.)

Although this article is strongly motivated by the physics, to the extent that we study only those mathematical properties which have been found to be useful, it is helpful to avoid reference to the applications. Much of the language is derived from the quantum mechanical vector-coupling problem in 3-space, but it may be more helpful to bear in mind operations on functions that span some other space, rather than physical vectors. To this end, we shall make much use of Dirac's ket-vector notation (Dirac 1930; Messiah 1965) and use greek letters, rather than j, to label group representations. This notation leads to a self-explanatory notation for the coupling (Wigner or Clebsch–Gordan) coefficients,

$$\langle \lambda_1 i_1;\, \lambda_2 i_2 | r\lambda i\rangle,$$

where r is a 'multiplicity label' which is required if the representation basis vector $|\lambda i\rangle$ occurs more than once in the Kronecker product $|\lambda_1 i_1\rangle\,|\lambda_2 i_2\rangle$. Derome & Sharp's (1965) notation for 3-jm symbols

$$(\lambda_1 \lambda_2 \lambda_3)_{r i_1 i_2 i_3}$$

is used for part A, but we change to a generalization of Wigner's notation (Wigner 1940)

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^r$$

for part B. The tensorial notation of Derome & Sharp is most useful for the analytic work of the first part but is less convenient in the second. Both notations suffer from disadvantages in interpretation because the position of the indices often bears little relation to the significance as matrix indices. The matrix elements of the unitary matrix representing group element R in representation λ will be written as

$$\lambda(R)_{ij}$$

rather than the more common notation

$$D^{\lambda}_{ij}(R).$$

The 1-jm symbol for this same representation will be denoted

$$(\lambda)_{ij}$$

and it is a unitary matrix of the same dimension as $\lambda(R)$. Complex conjugation is denoted by raising all indices

$$\{\lambda(R)_{ij}\}^* = \lambda(R)^{ij}.$$

We shall always sum over repeated indices, usually one lower and one upper. (Derome & Sharp (1965) use the notation $\lambda(R)^i{}_j$ for our $\lambda(R)_{ij}$, making this summation convention slightly more consistent, but this tends to confuse complex conjugates.) For the 3-jm symbol it is meaningful to raise and lower indices individually – this is the content of the Derome & Sharp lemma.

As stated above, we shall use the techniques of Derome & Sharp for manipulating j symbols, although Agrawala & Belinfante (1968) have given the generalization of the graphical techniques developed for R_3 by Jucys, Levinson & Vanagas (1962).

de Vries & van Zanten (1970) have recently suggested that some of the theory presented here in terms of representations can be given in terms of classes, at least for finite groups, and they suggest that one may be able to use this dual approach to extract additional information.

A. AN ARBITRARY GROUP

In this first part of the article we shall derive those properties of the various coefficients that are entirely independent of the basis chosen in the representation space. We restrict ourselves to representations of finite and of compact groups on a Hilbert space – since then only finite dimensional unitary representations occur. Most of the general results follow from these unitary properties, or from Schur's lemma, although in § 7 we use the techniques of character theory to obtain some particular results for certain groups – by way of illustration. In § 8 we do require a result from character theory, discussed elsewhere (Butler & King 1974). This result, on the occurrence of (pseudo-) symplectic representations in the product of (pseudo-) orthogonal representations, gives very important simplifications for all Lie groups and almost all finite groups. For the exceptions, one must retain certain phase factors of Derome & Sharp.

Let us begin by fixing our notation and reviewing some of the elementary properties of vector and matrix representations (see also Messiah 1965, appendix D).

2. Ket representations

Consider a Hilbert space. We use a Hilbert space but we only need a linear vector space with a non-degenerate positive definite scalar product defined on it, for we are then able to write an arbitrary element of the space in terms of any (complete) orthonormal basis (see, for example, Messiah 1965, p. 164). We use Dirac's bra–ket notation and for this paper we shall consider only elements of norm one in the space

$$|a\rangle, \ |b\rangle, \ \ldots$$

the scalar product of two vectors being written $\langle a|b\rangle$, which is a complex number, in general. Thus

$$\langle a|b\rangle = \langle b|a\rangle^* \tag{2.1}$$

and

$$\langle a|a\rangle = 1 \tag{2.2}$$

(* denotes complex conjugation, throughout).

Let a group G (elements $R, S, \ldots$) act on the Hilbert space unitarily so as to take all normal kets to normal kets, we write this as

$$O_R|a\rangle = |b\rangle. \tag{2.3}$$

We now use this group action to choose a basis on the Hilbert space. We first note, using only the group axioms, that this action splits the Hilbert space into a number of orthogonal spaces. We shall refer to these subspaces as representation spaces and label them by greek letters. We now choose an otherwise arbitrary orthonormal basis within each subspace. The collection of such bases is thus a complete basis for the Hilbert space. We denote an individual basis ket by three labels:

(1) an integer (latin letter) enumerating the basis within each representation space;

(2) a group-theoretic label (greek letter) for the representation space;

(3) a collection of convenient symbols to distinguish different subspaces with the same transformation properties. When this label is not relevant we shall often omit it. (For example, one-electron and two-electron wavefunctions are quite distinct (and orthogonal) but may well have the same transformation properties under the group of interest, say, rotations of an ion in a crystal.)

Thus a typical basis ket will be written

$$|x\lambda i\rangle \quad \text{or simply} \quad |\lambda i\rangle.$$

We shall denote the dimension of the representation space λ by $|\lambda|$. (In the standard notation of angular momentum theory the dimension of representation J is written $[J] = 2J+1$.) We also use the abbreviation

$$|\lambda_1, \lambda_2, \ldots| = |\lambda_1|\,|\lambda_2| \ldots.$$

The reduction of the Hilbert space into representation spaces will always be taken to be complete, and thus we will have only 'irreducible' representation spaces.

The action of the group element R on the basis ket $|\lambda i\rangle$ will be to give some ket $|a\rangle$, which may be written in terms of the basis of this representation space

$$\begin{aligned} O_R|x\lambda i\rangle &= |a\rangle \\ &= \sum_j |x\lambda j\rangle\, \lambda(R)_{ji}. \end{aligned} \tag{2.4}$$

Using (2.1) and (2.2) we have that the coefficients $\lambda(R)_{ji}$ form a unitary matrix. Thus, using the notation for the complex conjugate

$$\{\lambda(R)_{ij}\}^* = \lambda(R)^{ij} \tag{2.5}$$

and using the summation convention for repeated latin letters also mentioned in the introduction, we have

$$\lambda(R)_{ij}\lambda(R)^{i'j} = \delta_{ii'}, \tag{2.6}$$

and
$$\lambda(R)_{ij}\lambda(R)^{ij'} = \delta_{jj'}. \tag{2.7}$$

Acting on (2.4) with another element S of the group gives

$$\lambda(R)_{ij}\lambda(S)_{jk} = \lambda(RS)_{ik}, \tag{2.8}$$

which completes the statement that the matrices $\lambda(R)$ form an (irreducible) unitary matrix representation of the group. This is a representation in the usual sense of the word in that there is a homomorphism between the group elements R and the matrices $\lambda(R)$. In this sense the ket representation is not a representation at all, but it is usual to use the term, owing to the importance of (2.4).

In the above we said that we chose the basis arbitrarily, but we must make one restriction on this, namely if two or more representation spaces have similar transformation properties under the action of the group, we choose their bases in the same way. That is, the derived matrix representations are identical if the ket representations have similar transformation properties. Note that any linear combination of similar ket representations will also be similar.

3. Coupling (Wigner) coefficients

The tensor (Kronecker) product of two ket representations will give another ket representation, which will usually be reducible, the term $|x_1\lambda_1 i_1\rangle\,|x_2\lambda_2 i_2\rangle$ decomposing into components of each of the irreducible spaces of the product space:

$$|x_1\lambda_1 i_1\rangle\,|x_2\lambda_2 i_2\rangle = \sum_{r\lambda i} |(x_1\lambda_1, x_2\lambda_2)\, r\lambda i\rangle\,\langle r\lambda i|\lambda_1 i_1; \lambda_2 i_2\rangle. \tag{3.1}$$

The three kets $|x_1\lambda_1 i_1\rangle$, $|x_2\lambda_2 i_2\rangle$ and $|(x_1\lambda_1, x_2\lambda_2)\, r\lambda i\rangle$ may belong to three distinct Hilbert spaces or we may wish to consider them as part of the one Hilbert space that is the union of all three. For convenience, we have explicitly indicated that the ket $|(x_1\lambda_1, x_2\lambda_2)\, r\lambda i\rangle$ lies in the product space $(x_1\lambda_1, x_2\lambda_2)$. The label r is required whenever several linearly independent subspaces of the product space transform in a similar manner under the group. The separation according to this label is arbitrary in that any linear combination of similar representations can be chosen. It is trivial to verify that the kets $|r'\lambda i\rangle$ defined by

$$|r'\lambda i\rangle = \sum_r U^{\lambda}_{r'r}|r\lambda i\rangle \tag{3.2}$$

satisfy all our requirements of a suitable choice of basis, if U^{λ} is any unitary matrix. Even if there is no multiplicity ($r = 1$, only) there still remains an arbitrary phase for each λ. However, we must assume that the separation and phases are fixed – albeit arbitrarily. (For multiplicity free products, the phases are often chosen so that the coefficients of (3.1) are real and positive for i_1 and i maximum.)

The coefficients $\langle r\lambda i|\lambda_1 i_1;\, \lambda_2 i_2\rangle$ are known by many names, but for the reason given in the introduction, 'Clebsch–Gordan' is probably least justified, although most popular. We prefer the descriptive 'coupling coefficients'. Since the basis kets are orthonormal, it follows that the coupling coefficients form a unitary matrix of dimension $|\lambda_1|\,|\lambda_2|$, indices $(i_1 i_2)$ and $(r\lambda i)$

$$\sum_{i_1 i_2}\langle\lambda_1 i_1;\lambda_2 i_2|r\lambda i\rangle\langle r'\lambda' i'|\lambda_1 i_1;\,\lambda_2 i_2\rangle = \delta_{rr'}\delta_{\lambda\lambda'}\delta_{ii'}, \tag{3.3}$$

$$\sum_{r\lambda i}\langle\lambda_1 i_1;\,\lambda_2 i_2|r\lambda i\rangle\langle r\lambda i|\lambda_1 i_1';\,\lambda_2 i_2'\rangle = \delta_{i_1 i_1'}\delta_{i_2 i_2'}, \tag{3.4}$$

where we use the usual bra–ket notation for complex-conjugation (see (2.1)).

We can use the unitarity (3.3) in (3.1) in order to write the product ket in terms of the uncoupled kets

$$|(x_1\lambda_1, x_2\lambda_2)\, r\lambda i\rangle = \sum_{i_1 i_2}\langle\lambda_1 i_1;\lambda_2 i_2|r\lambda i\rangle\, |x_1\lambda_1 i_1\rangle\, |x_2\lambda_2 i_2\rangle. \tag{3.5}$$

These coupling coefficients also reduce the direct product of the representation matrices. Operating on (3.5) by a group operator, and then using (3.1), we may use the orthogonality of the product kets to give

$$\sum_{i_1 i_2 j_1 j_2}\langle r\lambda i|\lambda_1 i_1;\,\lambda_2 i_2\rangle\lambda_1(R)_{j_1 i_1}\lambda_2(R)_{j_2 i_2}\langle\lambda_1 j_1;\,\lambda_2 j_2|r'\lambda' i'\rangle = \delta_{rr'}\delta_{\lambda\lambda'}\lambda(R)_{i'i}, \tag{3.6}$$

which may be stated as 'The matrix of coupling coefficients is the matrix which reduces the direct product of two irreducible representation matrices into a direct sum of irreducible representation matrices, for all group elements'. Such a statement is often taken as the definition.

4. The complex-conjugate representation: the 1-*jm* symbol

Consider the complex conjugate of equation (2.8)

$$\lambda(R)^{ij}\lambda(S)^{jk} = \lambda(RS)^{ik} \quad \text{for all} \quad R, S \in G. \tag{4.1}$$

Hence these new matrices also form a matrix representation of the group, although possibly not in a standard basis. After being transformed to the standard basis, it is referred to as the representation complex conjugate to λ, and is denoted by λ^*. Clearly it is of the same dimension

$$|\lambda^*| = |\lambda| \tag{4.2}$$

and can easily be shown to be irreducible if λ is irreducible. If all the matrices are real (and thus orthogonal), then the operation is trivial, but in general two possibilities will occur. Either the representations are equivalent – that is, the matrices can be transformed into one another by a change of basis – or they are not. Even if λ and λ^* are not equivalent, a change of basis may still be required because in our choice of basis (2.4) nothing special was required. If the representations λ and λ^* are equivalent, then we have only the one standard basis so that

$$\lambda(R) = \lambda^*(R).$$

The necessary and sufficient condition for this equivalence is for the character χ^λ to be real. If the character is real, we may be able to choose a basis so that all the representation matrices $\lambda(R)$ are real. For this case the representation λ is said to be orthogonal, otherwise it is symplectic. If the character is complex, the representation is said to be complex.

We shall refer to the unitary matrix which performs the change of basis between $\lambda(R)^*$ and $\lambda^*(R)$ as the 1-*jm* symbol, (λ). With the notation

$$(\lambda)_{ij} = \{(\lambda)^{ij}\}^*, \tag{4.3}$$

then

$$\lambda^*(R)_{ij} = (\lambda)^{ki}\lambda(R)^{kl}(\lambda)_{lj}. \tag{4.4}$$

Note that there is an arbitrary phase in this definition, but one that is fixed by reference to the kets. In fact, taking complex conjugates twice gives

$$\lambda(R)_{in} = (\lambda)^{ij}(\lambda^*)_{jk}\lambda(R)_{kl}(\lambda^*)^{ml}(\lambda)_{nm}. \tag{4.5}$$

By Schur's lemma, the matrix $(\lambda)^{ij}(\lambda^*)_{jk}$ is a multiple ϕ_λ of the unit matrix δ_{ik} and from the unitarity of the 1-*jm* symbol this gives

$$(\lambda^*)_{ij} = \phi_\lambda(\lambda)_{ji} \quad \text{and} \quad \phi^*_\lambda = \phi_{\lambda^*}. \tag{4.6}$$

We refer to ϕ_λ as the 1-*j* phase.

Example for the group R_3, all the characters are real, but even for the orthogonal representations, the basis is not chosen so that the representation matrices are real. The basis is chosen so that

$$Y^*_{jm} = (-)^m Y_{j-m}.$$

We choose the overall phase to give real 1-*jm* symbols,

$$(j)_{mm'} = (-)^{j-m}\delta_{m-m'} \quad \text{and} \quad \phi_j = (-)^{2j}.$$

In § 6 we shall show that for real characters, the 1-*j* phase ϕ_λ is fixed by group theory as ± 1, and can be found using character theory only. On the other hand, the 1-*jm* symbol is clearly dependent upon the choice of basis. It is for this reason the label m is included in its name.

5. The 3-*jm* symbol

We are now in a position to define the 3-*jm* symbol by

$$\langle r\lambda i|\lambda_1 i_1;\lambda_2 i_2\rangle = |\lambda|^{\frac{1}{2}} K(\lambda_1\lambda_2\lambda)_{rs}(\lambda)^{ij}(\lambda_1\lambda_2\lambda^*)_{s i_1 i_2 j} \tag{5.1}$$

(a sum over s and j is implied). One inserts the dimensionality factor $|\lambda|^{\frac{1}{2}}$ and the 1-*jm* symbol $(\lambda)^{ij}$ to put the three representations λ_1, λ_2 and λ^* on an equal footing, the precise reasons for this will become apparent in the algebra which follows. The unitary matrix $K(\lambda_1\lambda_2\lambda)$ in the multiplicity label is inserted for historical reasons. We shall assume that the separation in r in the coupling coefficient is arbitrary, but fixed, but that K is to be adjusted to produce the simplest possible properties for the 3-*jm* symbol. If we have no accepted choice for the coupling coefficient's phase we could redefine the coupling coefficient by

$$\langle r\lambda i|\lambda_1 i_1;\lambda_2 i_2\rangle_{\text{sensible}} = |\lambda|^{\frac{1}{2}}(\lambda)^{ij}(\lambda_1\lambda_2\lambda^*)_{r\,i_1 i_2 j}. \tag{5.2}$$

Hamermesh (1962, p. 376) suggests this, but the Condon & Shortley (1935) phases give for R_3 (where $r = 1$ only)

$$K_{\text{C.S.}}(j_1 j_2 j) = (-)^{j_1-j_2+j}$$

and considerable confusion would be created by attempting to redefine phases now. The insertion of the matrix K also serves the purpose of reminding readers of the essential arbitrariness in the multiplicity index.

It follows from the definition (3.1) of the coupling coefficients, and (5.1) above, that both coefficients are identically zero wherever the representation λ does not occur in the Kronecker product $\lambda_1\times\lambda_2$. This information was omitted from (3.3) and (3.4) and for simplicity we shall continue not making explicit reference to it.

The 3-*jm* symbol is usually written in the Wigner notation, which we shall use ourselves in the later part of this article,

$$(\lambda_1\lambda_2\lambda_3)_{r\,i_1 i_2 i_3} = \begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ i_1 & i_2 & i_3\end{pmatrix}^{r}, \tag{5.3}$$

and the complex conjugate symbols are denoted by

$$(\lambda_1\lambda_2\lambda_3)^{r\,i_1 i_2 i_3} = \begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ i_1 & i_2 & i_3\end{pmatrix}^{*r}. \tag{5.4}$$

The unitary relations follow directly from the definition, and the unitarity of the other terms,

$$|\lambda_3|\,(\lambda_1\lambda_2\lambda_3)^{r\,i_1 i_2 i_3}(\lambda_1\lambda_2\lambda_3')_{r'\,i_1 i_2 i_3'} = \delta_{i_3 i_3'}\delta_{\lambda_3\lambda_3'}\delta_{rr'}, \tag{5.5}$$

and

$$\sum_{\lambda_3}|\lambda_3|\,(\lambda_1\lambda_2\lambda_3)^{r\,i_1 i_2 i_3}(\lambda_1\lambda_2\lambda_3)_{r\,i_1' i_2' i_3} = \delta_{i_1 i_1'}\delta_{i_2 i_2'}. \tag{5.6}$$

These equations are very similar to those for the coupling coefficients, but the 1-*jm* symbol in (5.1) makes its presence felt by transforming the product representation in (3.6) to its complex conjugate. By using (5.6) to rearrange terms, (3.6) becomes

$$(\lambda_1\lambda_2\lambda_3)_{r\,i_1 i_2 i_3}\lambda_1(R)_{i_1 j_1}\lambda_2(R)_{i_2 j_2} = \lambda_3(R)^{i_3 j_3}(\lambda_1\lambda_2\lambda_3)_{r\,j_1 j_2 j_3}. \tag{5.7}$$

This equation has been used by Hamermesh (1962, equations 7–186) as the starting point for a calculation of the numerical values of the coefficients for the symmetric groups.

We could use the unitarity of $\lambda_3(R)$ to shift it to the left here, to give a very symmetric expression, but we leave the exploration of such symmetries to the next section. Instead, we use (5.6) again, to give Derome & Sharp's defining equation (1965),

$$\lambda_1(R)_{i_1 j_1}\lambda_2(R)_{i_2 j_2} = \sum_{\lambda_3}|\lambda_3|\,(\lambda_1\lambda_2\lambda_3)^{r\,i_1 i_2 i_3}\lambda_3(R)^{i_3 j_3}(\lambda_1\lambda_2\lambda_3)_{r\,j_1 j_2 j_3}. \tag{5.8}$$

This definition, like ours of (5.1), is incomplete, being undefined up to any transformation of the multiplicity label. They are able to define the 1-*jm* symbol as a special case of (5.8), when λ_1 is the identity representation – which we shall denote by 1. The matrices $\lambda_1(R)$ are simply the number 1, and the sum over the product representations reduces to one term:

$$\lambda_2(R)_{i_2j_2} = |\lambda_3|\,(1\lambda_2\lambda_3)^{11\,i_2i_3}\lambda_3(R)^{i_3j_3}(1\lambda_2\lambda_3)_{11\,j_2j_3}. \tag{5.9}$$

This is to be compared with the definition of the 1-*jm* (4.4), thus†

$$(\lambda)_{ij} = |\lambda|^{\frac{1}{2}}(1\lambda\lambda^*)_{11\,ij} \tag{5.10}$$

Now there is another unitarity condition on the representation matrices, a consequence of the finiteness (or compactness) of the group,

$$\frac{1}{g}\int_G \lambda(R)_{ij}\lambda'(R)^{i'j'}\,\mathrm{d}R = \delta_{\lambda\lambda'}\delta_{ii'}\delta_{jj'}, \tag{5.11}$$

where
$$g = \int_G \mathrm{d}R$$

is the integral (or sum, for a finite group) carried out over all group elements. Using this in (5.8) gives

$$\frac{1}{g}\int \lambda_1(R)_{i_1j_1}\lambda_2(R)_{i_2j_2}\lambda_3(R)_{i_3j_3}\,\mathrm{d}R = (\lambda_1\lambda_2\lambda_3)^{r\,i_1i_2i_3}(\lambda_1\lambda_2\lambda_3)_{r\,j_1j_2j_3}. \tag{5.12}$$

Derome & Sharp use this result extensively to prove their results and we follow them by first deriving the symmetries of the 3-*jm* symbols under permutations of λ_1, λ_2 and λ_3.

6. Reordering symmetries: the 3-*j* permutation matrices

The left side of (5.12) is independent of the order of the product λ_1, λ_2 and λ_3 (and incidentally – group theory shows that it is zero if the triple product does not contain the identity representation). Thus we must have

$$(\lambda_1\lambda_2\lambda_3)^{r\,i_1i_2i_3}(\lambda_1\lambda_2\lambda_3)_{r\,j_1j_2j_3} = (\lambda_a\lambda_b\lambda_c)^{s\,i_ai_bi_c}(\lambda_a\lambda_b\lambda_c)_{s\,j_aj_bj_c}, \tag{6.1}$$

where $a\,b\,c$ is any permutation, π, of 1 2 3. It follows that

$$(\lambda_a\lambda_b\lambda_c)_{s\,i_ai_bi_c} = m(\pi,\lambda_1\lambda_2\lambda_3)_{sr}(\lambda_1\lambda_2\lambda_3)_{r\,i_1i_2i_3}, \tag{6.2}$$

where m is a unitary matrix. In this section we shall quote and extend Derome's (1966) results on the possible structures of the 3-*j* permutation matrices $m(\pi)$.

Clearly the matrices $m(\pi)$ exist for all possible reorderings but will not all be independent. We refer the reader to Derome for a detailed consideration of their arbitrarity, only sketching the argument here. We use the freedom in the matrices $K(\lambda_1\lambda_2\lambda_3^*)$ of (5.1) to adjust the relative definitions of the various 3-*jm* symbols in the various orderings, in order to give the simplest possible matrices m. First, it is clear that if λ_1, λ_2 and λ_3 are all different, we shall have six independent choices for K, and they could be chosen to give almost any m; on the other hand, if the three representations are equal, then only one K is available, and group theory will impose its greatest restriction. In this case, we will also have fewer m's since then $m(\pi,\lambda_1\lambda_2\lambda_3) = m(\pi,\lambda_2\lambda_1\lambda_3)$,

† Strictly, Derome & Sharp set the representation λ_2 equal the identity but then their phase for the 6-*j* symbols differs slightly from the standard one. If Derome & Sharp's (1965) equation (3.1) is changed to our (5.10), then no other changes need be made and we obtain the standard phases for the R_3 6*j*. Thus in (8.1) and (9.6) we shall follow Derome & Sharp's notation, which unfortunately differs from that of Wigner (1940). In (9.6) we reorder some subscripts relative to Derome & Sharp.

etc. (Our first choice will be to always ask this.) There will also be restrictions given by complex conjugation of the equations and representations.

The three possibilities for equalities among the representations give the three cases

(*a*) if $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, almost complete freedom exists and we choose either

$$m(\pi)_{rs} = \delta_{rs} \tag{6.3}$$

or

$$m(\mathrm{i}, \lambda_1\lambda_2\lambda_3)_{rs} = \theta(\lambda_1\lambda_2\lambda_3 r)\,\delta_{rs} \quad \text{and} \quad m(\mathrm{c})_{rs} = \delta_{rs}, \tag{6.4}$$

where $\theta(\lambda_1\lambda_2\lambda_3 r) = \pm 1$, where we use the labels $\pi = \mathrm{i}$ for odd permutations (interchanges) and $\pi = \mathrm{c}$ for even (cyclic) permutations. The first choice is simpler in one sense, but sometimes there are reasons for the second. We discuss this later in this section and in § 8.

(*b*) if $\lambda_1 = \lambda_2 \neq \lambda_3$, it is still always possible to make a diagonal choice, but a phase $\theta(\lambda\lambda\lambda' r) = \pm 1$ enters of necessity, depending on whether the rth term of λ'^* occurs in the symmetric or antisymmetric part of the product $\lambda \times \lambda$. However, we choose the K's so that

$$m(\mathrm{i}, \lambda\lambda\lambda')_{rs} = \theta(\lambda\lambda\lambda' r)\,\delta_{rs}, \tag{6.5a}$$

giving

$$m(\mathrm{c}, \lambda\lambda\lambda')_{rs} = \delta_{rs}. \tag{6.5b}$$

We note that (6.5) is not the choice of Hamermesh (1962, equations 7–206*c*) since he does not choose

$$m(\mathrm{i}, \lambda\lambda\lambda')_{rs} = m(\mathrm{i}, \lambda\lambda'\lambda)_{rs}.$$

(*c*) if $\lambda_1 = \lambda_2 = \lambda_3$, Derome proves that it may be impossible to choose the one K in such a way that all $m(\pi)$ are diagonal. The triple product $\lambda \times \lambda \times \lambda$ may contain the identity representation, in the symmetric, antisymmetric or mixed symmetry part, or even several times in each. This result was also implied by Griffith (1960). We choose K to separate the three types of symmetry, labelling the symmetries with Young diagrams for S_3, namely [3], [21] and [1^3].

$$m(\pi) = \begin{bmatrix} m^{[3]}(\pi) & 0 & 0 \\ 0 & m^{[21]}(\pi) & 0 \\ 0 & 0 & m^{[1^3]}(\pi) \end{bmatrix}. \tag{6.6a}$$

For the fully symmetric [3] and fully antisymmetric [1^3] terms, the submatrices are as for example (*b*) above. The mixed symmetry terms [21] occur in pairs and we could thus take permutation submatrices of dimension 2, taking K so they will be in standard form for this representation of S_3, namely (Hamermesh 1962, p. 224)

$$m((12)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad m((23)) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}, \quad m((13)) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix},$$

$$m((123)) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}, \quad m((132)) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}. \tag{6.6b}$$

(The notation should be self-explanatory, but for the cyclic permutations it is essential to note carefully the definition (6.2) and not to confuse the action of $m((123))$ and $m((132))$.) However, so as not to lose the ordering of $m((12))$

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},$$

we choose to order the mixed symmetry block so that $m^{[21]}((23))$, for example, is

$$\begin{bmatrix} -\frac{1}{2}I & \frac{1}{2}\sqrt{3}\,I \\ \frac{1}{2}\sqrt{3}\,I & \frac{1}{2}I \end{bmatrix}, \tag{6.6c}$$

where I is the unit matrix of dimension equal to the number of mixed symmetry pairs.

For simply reducible groups, the 3-j permutation matrices are always of dimension one – by definition – and thus (6.6) is not required. Indeed the situation simplifies further in that the phases of interchanges in both (*b*) and (*c*) can be written uniquely in the form $(-)^{\lambda_1+\lambda_2+\lambda_3}$, where $(-)^{\lambda}$ is a phase permanently associated with representation λ. For such groups it is convenient to choose this same phase factor for case (*a*).

Note that since the 1-*jm* symbol is a special case of the 3-*jm* symbol, we have

$$\phi_\lambda = \phi_{\lambda^*} = m(\mathrm{i}, \lambda\lambda^*1)_{11} = \theta(\lambda\lambda^*11) = \pm 1 \tag{6.7}$$

(for R_3 it follows that $\phi_j = (-)^{2j}$).

Choices (6.3)–(6.6) give the permutation matrices explicitly. In the next section we show how to use character theory to evaluate the symmetry types. From character theory, it is clear that the symmetries associated with the triple product $\lambda_1^* \times \lambda_2^* \times \lambda_3^*$ are the same as for $\lambda_1 \times \lambda_2 \times \lambda_3$. Thus simply ordering the symmetries appropriately – which would be natural anyway – gives

$$m(\pi, \lambda_1\lambda_2\lambda_3)_{rs} = m(\pi, \lambda_1^*\lambda_2^*\lambda_3^*)_{rs}. \tag{6.8}$$

Note also that all the choices including (*c*), give

$$m((12), \lambda_1\lambda_2\lambda_3)_{rs} = \theta(\lambda_1\lambda_2\lambda_3 r)\,\delta_{rs} \tag{6.9}$$

and that m is always a real (orthogonal) matrix.

7. Examples of permutation symmetries

In the previous section we gave Derome's result that the modulus of the 3-*jm* symbol is not always invariant under permutations of the representations. Expressed differently, the 3-j permutation matrix cannot always be chosen diagonal. In particular, the Kronecker cube of a representation may contain the scalar representation with mixed symmetry, and this leads to a two dimensional submatrix on the diagonal of the full 3-j permutation matrix. We shall use character theory to investigate such occurrences for various groups, refering to representations whose Kronecker cube does not contain such a scalar, as 'simple phase' representations. Groups for which all representations are simple phase are thus described as being simple phase groups (van Zanten & de Vries 1973).

Since the various symmetry properties of the 3-*jm* symbols depend only on the properties of the representations as a whole, they can be calculated by the techniques of character theory (Hamermesh 1962; Robinson 1961). In particular, the number x of scalars of type [21] in the cube of a representation is given by

$$x = \frac{1}{g}\sum_R ((\chi^\lambda(R))^3 - \chi^\lambda(R^3)). \tag{7.1}$$

The algebra of *S*-functions (Littlewood 1950; Robinson 1961; Wybourne 1970) contains an operation (plethysm) which enables the separation of the symmetries to be evaluated without recourse to character tables. We have the notation, (*a*) the symmetric and antisymmetric terms of the Kronecker square of λ, are the symmetrized Kronecker products $\lambda \otimes \{2\}$ and $\lambda \otimes \{1^2\}$ respectively, and

(*b*) the fully symmetric, mixed symmetry and fully antisymmetric terms of the Kronecker cube are written $\lambda \otimes \{3\}$, $\lambda \otimes \{21\}$ and $\lambda \otimes \{1^3\}$ respectively.

These symmetrized products on the representations λ of the group G may be stated as 'constructing irreducible representations of S_2 (or S_3) out of the representation matrices of G. These induced representations of S_2 (or S_3) remain representations of G, and as such may be expressed as a direct sum of representations of G.' The operation is thus related also to the operation of wreath products (Kerber 1973). The evaluation of symmetrized products using the plethysm of S-functions is largely solved but it is not the purpose of this article to review this, so we refer the reader to Wybourne (1970) and various recent articles (Butler & Wybourne 1971; Butler & King 1973–4) for discussions of the problem and for earlier references. We simply give the results of some simple calculations (Butler & King 1974).

The representation [321] of the symmetric group S_6 is the first really interesting case for the symmetric groups. Its square contains the representation $[41^2]$ once symmetrically and three times antisymmetrically. Its cube contains the identity twice fully symmetrically, one pair with mixed symmetry, and once antisymmetrically. [321] of S_6 was the example given by Derome (1966) of a non-simple phase representation. It is the smallest such representation for the symmetric groups. van Zanten & de Vries give many other finite groups that are not simple phase noting, in particular, smaller groups (van Zanten & de Vries 1973).

There are infinitely many such non-simple phase representations for most Lie groups. For the orthogonal, rotation and symplectic groups (O_n, R_n and Sp_n), the first such representations are

O_n and R_n	[21]	$n = 5$ only
	[31]	$n \geqslant 5$
	[211]	$n \geqslant 7$
Sp_n	$\langle 31 \rangle$	$n \geqslant 4$
	$\langle 211 \rangle$	$n \geqslant 6$

The groups $R_4 \simeq R_3 \times R_3$, $R_3 \simeq Sp_2$ and R_2 are simply reducible. For the unitary groups, the plethysms involved are more difficult to evaluate and general results not so readily stated unless we use the composite-tableau (back-to-back) notation (Littlewood 1943; Wybourne 1970; Abramsky & King 1970; King 1970). In this notation, the representation of SU_n usually denoted by the partition $\{\lambda\}$ is denoted by the pair $\{\mu; \nu\}$ where

$$\mu_i + r = \lambda_i \quad \text{and} \quad r - \nu_i = \lambda_{n-i} \quad \text{for} \quad i \leqslant \tfrac{1}{2}n.$$

Using this, we have that the representations are usually not simple phase for all n whenever μ and ν are partitions of the same number. Some of the simplest non-simple phase representations are the following:

$$SU_n\ \{21; 21\}, \quad \{31; 31\}, \quad \{41; 32\}, \quad \{32; 41\} \quad (n \geqslant 4),$$

e.g. for SU_4 {4310}, {6420}, {7410} and {7630}. The representation {7410} of SU_4 is interesting in that it is one of the smallest non-simple phase representations for SU_n that is also not real,

$$\{7410\}^* = \{7630\}.$$

The isomorphism between the representations of SU_4 and R_6 furnishes a similar example for the rotation groups

$$R_6[521]^* = [52-1].$$

(All characters of the symmetric, symplectic and orthogonal groups are real, also the rotation groups except the sequence R_2 R_6 R_{10} ....)

By means of these techniques, it is easy to show that SU_3, although not multiplicity free, is in fact a simple phase group. Derome obtained this result by direct integration (Derome 1967). S_5 is another such group.

8. Raising and lowering indices

We shall now investigate the relation between the 3-*jm* symbol for the triple $\lambda_1\lambda_2\lambda_3$ and the triple $\lambda_1^*\lambda_2^*\lambda_3^*$. The first result is the Derome & Sharp lemma and is the key to the usefulness of their tensorial notation. We prove a theorem that further simplifies their result. Their result follows quite simply from (5.12) and the property of the 1-*jm* symbol of transforming the representation matrices to their complex conjugates. First we formally write

$$(\lambda_1)^{i_1j_1}(\lambda_1^*\lambda_2\lambda_3)_{r\,j_1i_2i_3} = (\lambda_1\lambda_2\lambda_3)_r{}^{i_1}{}_{i_2i_3} \tag{8.1}$$

and raise the indices i_2 and i_3 in a similar fashion. The Derome & Sharp lemma then states that there exists a unitary matrix A, dependent only on the product as a whole and independent of the basis $i_1i_2i_3$, with the property that

$$A(\lambda_1\lambda_2\lambda_3)^{rs}(\lambda_1\lambda_2\lambda_3)_s{}^{i_1i_2i_3} = (\lambda_1\lambda_2\lambda_3)^{r\,i_1i_2i_3}, \tag{8.2}$$

that is, the complex conjugated 3-*jm* symbol is equal (after independent changes of the basis of each of the kets and of the multiplicity label) to the 3-*jm* symbol for the complex conjugate representations. Written out in full in the Wigner notation, (8.2) reads

$$\sum_{sj_1j_2j_3} A(\lambda_1\lambda_2\lambda_3)^{rs}(\lambda_1)^{i_1j_1}(\lambda_2)^{i_2j_2}(\lambda_3)^{i_3j_3}\begin{pmatrix}\lambda_1^* & \lambda_2^* & \lambda_3^*\\ j_1 & j_2 & j_3\end{pmatrix}^s = \begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ i_1 & i_2 & i_3\end{pmatrix}^{*r}. \tag{8.3}$$

This lemma follows from the integral (5.12). For the group R_3, the characters are real and the 1-*jm* symbols known, leading immediately to

$$A(j_1j_2j_3) = 1,$$

and

$$\begin{pmatrix}j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix} = (-)^{j_1+j_2+j_3}\begin{pmatrix}j_1 & j_2 & j_3\\ -m_1 & -m_2 & -m_3\end{pmatrix}.$$

Let us consider the properties of $A(\lambda_1\lambda_2\lambda_3)$ in the general case. We first use the properties of the 3-*j* permutation matrices found or fixed in § 6. Consider the case of simple phase products first. We always have

$$(\lambda_1\lambda_2\lambda_3)_{r\,i_1i_2i_3} = \theta_r(\lambda_a\lambda_b\lambda_c)_{r\,i_ai_bi_c}, \tag{8.4}$$

and

$$(\lambda_1^*\lambda_2^*\lambda_3^*)_{r\,j_1j_2j_3} = \theta_r(\lambda_a^*\lambda_b^*\lambda_c^*)_{r\,j_aj_bj_c}, \tag{8.5}$$

where $\theta_r = \pm 1$ and we have used the result (6.8) that the products $\lambda_1\times\lambda_2\times\lambda_3$ and $\lambda_1^*\times\lambda_2^*\times\lambda_3^*$ have the same symmetry types even when they are distinct. Comparing the result of raising the unpermuted 3-*jm* symbol and then permuting, with raising the permuted 3-*jm*, gives

$$A(\lambda_1\lambda_2\lambda_3)_{rs}\theta_r = A(\lambda_a\lambda_b\lambda_c)_{rs}\theta_s. \tag{8.6}$$

Thus

$$A(\lambda_1\lambda_2\lambda_3)_{rs} = 0 \quad \text{if} \quad \text{symmetry } r \neq \text{symmetry } s, \tag{8.7}$$

otherwise

$$A(\lambda_1\lambda_2\lambda_3)_{rs} = A(\lambda_a\lambda_b\lambda_c)_{rs}. \tag{8.8}$$

(8.8) clearly holds for the non-simple phase case since $\lambda_1 = \lambda_2 = \lambda_3$ and it is straightforward to show (8.7) generalizes also. We have shown, therefore, that the block diagonalization of the multiplicity spaces by symmetry type, has also block diagonalized the transformations between pairs of complex conjugate multiplicity spaces.

Next, comparing the action of $A(\lambda_1^*\lambda_2^*\lambda_3^*)$ with $A(\lambda_1\lambda_2\lambda_3)$ gives that

$$A(\lambda_1^*\lambda_2^*\lambda_3^*)_{rs} = \phi_{\lambda_1}\phi_{\lambda_2}\phi_{\lambda_3}A(\lambda_1\lambda_2\lambda_3)_{sr}. \tag{8.9}$$

For R_3 it is well known that the product of the three 1-j phases is always $+1$ when $\lambda_1\lambda_2\lambda_3$ form a triple, but this is by no means obvious in general. Indeed the choice of $\phi_\lambda = 1$ when $\lambda \neq \lambda^*$, made by both Derome & Sharp (1965) and Agrawala & Belinfante (1968) but not used by either, soon leads to a negative sign in (8.9) (e.g. $\{1^3\}\{1^2\}\{1^1\}$ of SU_6, since $\phi_{\{1^3\}} = -1$). Elsewhere (Butler & King 1974) we show that it is nearly always possible to choose the 1-j phases so that

$$\phi_{\lambda_1}\phi_{\lambda_2}\phi_{\lambda_3} = 1. \tag{8.10}$$

One requires both a knowledge of ϕ_λ, $\lambda^* = \lambda$ and of the Kronecker product rules, in order to prove this. The investigations show that (8.10) breaks down for very few finite groups only, the smallest such group being the group of order 24 with generators related by $S^2 = T^2 = (ST)^3$ and labelled $\langle -2, 2, 3\rangle$ in the notation of Coxeter & Moser (Biedenharn, Brouwer & Sharp 1968). The above group of order 24 can be made to satisfy (8.10) by using complex 1-j phases, but such a choice would introduce many complications in the 3-j permutation matrices, as these would no longer be all real. The choice of ϕ_λ corresponds to asking for a particular relation between the 1-jm symbols $(\lambda)_{ij}$ and $(\lambda^*)_{ji}$, so that the phases relating the 3-jm symbols cancel without the insertion of an additional phase.

To prove (8.10) we must not only show that it holds when there is no choice (when $\lambda_1 = \lambda_1^*$, etc.) but also that there is an appropriate choice when $\lambda \neq \lambda^*$. For example, while most unitary groups have $\phi_\lambda = 1$ if $\lambda = \lambda^*$, some do not. Those of the series $SU(4k+2)$ form the only exceptions, containing representations where $\phi_\lambda = -1$. For representations with real characters the 1-j phase is given by

$$\text{for} \quad SU(4k+2) \quad \phi_\lambda = (-)^l, \quad \lambda \text{ a partition of } l. \tag{8.11}$$

By using the Littlewood Richardson rule, it is easy to verify that if $\lambda_1 \times \lambda_2 \times \lambda_3$ contains the identity, then $l_1 + l_2 + l_3$ is even. Thus choosing (8.11) when $\lambda \neq \lambda^*$ will ensure that (8.10) is always satisfied. However, such considerations do not fix the phase for $SU(4k)$. Either $(\pm)^l$ will satisfy both requirements.

Suitable choices of the 1-j phase for the semi-simple Lie groups are given in table 1, p. 561.

Now we consider the freedom in the matrix K of (5.1), and use the results (8.7)–(8.10) to prove that it is always possible within the limits of the above mentioned validity of (8.10) to arrange matters so that $A(\lambda_1\lambda_2\lambda_3)_{rs} = \delta_{rs}$. This is equivalent to saying that the position of the multiplicity index in the 3-jm symbol is not significant. This will lead to many simplifications in the properties of 6-j and 9-j symbols.

THEOREM: *The multiplicity metric tensor A_{rs}, can always be chosen the unit matrix (for groups that (8.10) holds).*

We have shown that the matrix $A(\lambda_1\lambda_2\lambda_3)_{rs}$ is block diagonalized with regard to the symmetry types of the triple $(\lambda_1\lambda_2\lambda_3)$. Let us consider changes in the separation in the multiplicity index

$$U(\lambda_1\lambda_2\lambda_3)_{r'r}(\lambda_1\lambda_2\lambda_3)_{r\,i_1i_2i_3} = (\lambda_1\lambda_2\lambda_3)'_{r'\,i_1i_2i_3}. \tag{8.12}$$

From the requirement that U does not change the permutation matrices, we have that U is block diagonal with respect to symmetry type, and that

$$U(\lambda_1\lambda_2\lambda_3)_{r'r} = U(\pi(\lambda_1\lambda_2\lambda_3))_{r'r}, \tag{8.13}$$

but is otherwise arbitrary.

Such a transformation induces a change in A,

$$A(\lambda_1\lambda_2\lambda_3)'_{r's} = U(\lambda_1\lambda_2\lambda_3)_{r'r}\,A(\lambda_1\lambda_2\lambda_3)_{rs}\,U(\lambda_1^*\lambda_2^*\lambda_3^*)_{s's}. \tag{8.14}$$

We must consider two cases: first if the space $\lambda_1^*\lambda_2^*\lambda_3^*$ is the same as the space $\lambda_1\lambda_2\lambda_3$ (or some permutation of it). For this case we are required to choose

$$U(\lambda_1^*\lambda_2^*\lambda_3^*)_{r'r} = U(\lambda_1\lambda_2\lambda_3)_{r'r} \tag{8.15}$$

and the transformation (A 3) may be written

$$A' = UAU^T, \tag{8.16}$$

where we know that A is symmetric ((8.8) and (8.9)). Now lemma 2 of Gantmacher (1960, p. 4) states that if A is unitary and symmetric, it may be written in the form

$$A = e^{iS} \quad \text{where} \quad S = S^* = S^T. \tag{8.17}$$

Taking

$$U = e^{-\frac{1}{2}iS}$$

gives

$$A' = e^{-\frac{1}{2}iS}e^{iS}e^{-\frac{1}{2}iS} = I. \tag{8.18}$$

The second case is simpler, if $\lambda_1\lambda_2\lambda_3$ and $\lambda_1^*\lambda_2^*\lambda_3^*$ are different spaces, $U(\lambda_1\lambda_2\lambda_3)$ and $U(\lambda_1^*\lambda_2^*\lambda_3^*)$ may be chosen independently. Taking

$$U(\lambda_1\lambda_2\lambda_3) = I \quad \text{and} \quad U(\lambda_1^*\lambda_2^*\lambda_3^*) = A^*$$

gives

$$A' = I. \tag{8.19}$$

TABLE 1. CHOICES OF THE 1-j PHASE SATISFYING EQUATION (8.10), FOR THE LIE GROUPS

SU_{2k+1}	$\phi_\lambda = 1$			
SU_{4k}	$\phi_\lambda = 1$ or $(-)^l$			
SU_{4k+2}	$\phi_\lambda = (-)^l$			
R_n	$n = 0, 1, 7$ (mod 8)	$\phi = 1$		
	$n = 2, 6$ (mod 8)	$\phi_{\text{true}} = 1$	$\phi_{\text{spin}} = \pm 1$	
	$n = 3, 4, 5$ (mod 8)	$\phi_{\text{true}} = 1$	$\phi_{\text{spin}} = -1$	
Sp_n	$\phi_\lambda = (-)^l$			
G_2 F_4 and E_8	$\phi_\lambda = 1$			
E_6	$\phi_{\text{true}} = 1$	$\phi_{\text{spin}} = \pm 1$		
E_7	$\phi_{\text{true}} = 1$	$\phi_{\text{spin}} = -1$		

In the remainder of this article we shall assume (8.10) holds, for although the general case can be readily treated, it seems unlikely that it will be needed by physicists.

Before leaving our discussion of the properties of the 3-jm symbols it is worth remarking on the constraints we have placed upon K – the separation of the multiplicities. First, in the separation according to symmetry type, we required only that the symmetric, mixed, and antisymmetric groups of terms were separated, any unitary linear combination of terms of the same type was allowed – except for mixed symmetry terms, when the phase between the appropriate pairs was fixed. The separation for one product $\lambda_1 \times \lambda_2 \supset \lambda$ was then chosen for all permutations of $\lambda_1 \lambda_2 \lambda^*$. These choices gave the simplest form to the 3-j permutation matrices. Second, A, the unit matrix, required a close relation between the triple products $\lambda_1 \lambda_2 \lambda_3$ and $\lambda_1^* \lambda_2^* \lambda_3^*$. If these triples are different any unitary linear combination of one induces the complex conjugate transformation in the other. If the triples are the same (up to a permutation) then we are restricted to allowing real (orthogonal) transformations only.

These choices can be illustrated by choosing two extreme triples with multiplicity four. If $\lambda_1 \lambda_1^* \lambda_2 \lambda_2^* \lambda_3$ and λ_3^* are all different, then we have no restrictions of $K(\lambda_1 \lambda_2 \lambda_3)$, and any unitary linear combination of one solution will also be a solution, but by the choices for $m(\pi, \lambda_1 \lambda_2 \lambda_3)$ and $A(\lambda_1 \lambda_2 \lambda_3)$ we have fixed eleven other separations – the five other orderings, and the six complex conjugate triples. On the other hand, if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_1^*$, and also we have one symmetric, one pair of mixed symmetry and one antisymmetric term in the product, the only remaining freedom in K is a choice of three signs (± 1), one for each type of symmetry, but here there is only one K anyway.

9. The recoupling (Racah) coefficient: the 6-j symbol

It is the purpose of this and the following section to define the various recoupling coefficients and n–j symbols which arise when Kronecker products of more than two ket-representations are considered. Although in one sense no new ideas are introduced, the coefficients are fundamentally different from the 1-jm and 3-jm symbols in that these j symbols depend on sums of the former in such a way that they are completely independent of the choice of basis. They are not, however, independent of the separation of the multiplicities of the product, indeed they are much better considered as generalizations of the 1-j phase and 3-j permutation matrices.

Consider the direct product of three ket-representations

$$|\lambda_1 i_1\rangle |\lambda_2 i_2\rangle |\lambda_3 i_3\rangle. \tag{9.1}$$

Using coupling coefficients and the methods of § 3 we reduce this ket belonging to the product space $(\lambda_1 \lambda_2 \lambda_3)$ by a two stage process, and by two distinct routes. Either

$$\begin{aligned}|\lambda_1 i_1, \lambda_2 i_2, \lambda_3 i_3\rangle &= \sum_{r_{12}\lambda_{12}i_{12}} \langle r_{12}\lambda_{12}i_{12}|\lambda_1 i_1; \lambda_2 i_2\rangle \,|(\lambda_1\lambda_2)\, r_{12}\lambda_{12}i_{12}, \lambda_3 i_3\rangle \\ &= \sum_{r_{12}\lambda_{12}i_{12}r\lambda i} \langle r_{12}\lambda_{12}i_{12}|\lambda_1 i_1; \lambda_2 i_2\rangle \langle r\lambda i|\lambda_{12}i_{12}; \lambda_3 i_3\rangle \,|(\lambda_1\lambda_2)\, r_{12}\lambda_{12}, \lambda_3, r\lambda i\rangle, \end{aligned} \tag{9.2}$$

or

$$\begin{aligned}|\lambda_1 i_1, \lambda_2 i_2, \lambda_3 i_3\rangle &= \sum_{r_{23}\lambda_{23}i_{23}} \langle r_{23}\lambda_{23}i_{23}|\lambda_2 i_2; \lambda_3 i_3\rangle \,|\lambda_1 i_1, (\lambda_2\lambda_3)\, r_{23}\lambda_{23}i_{23}\rangle \\ &= \sum_{r_{23}\lambda_{23}i_{23}s\mu j} \langle r_{23}\lambda_{23}i_{23}|\lambda_2 i_2; \lambda_3 i_3\rangle \langle s\mu j|\lambda_1 i_1; \lambda_{23}i_{23}\rangle \,|\lambda_1(\lambda_2\lambda_3)\, r_{23}\lambda_{23}, s\mu j\rangle. \end{aligned} \tag{9.3}$$

The ket-representation spaces of (9.2) and (9.3) are similar but the bases are different. There is thus a unitary matrix which performs the change of basis between (9.2) and (9.3), but it is restricted by (3.2) to being diagonal in the representation label ($\lambda = \mu$) and independent of $i = j$.

These recoupling coefficients are also called overlap integrals or Racah coefficients and may be defined by

$$|(\lambda_1\lambda_2)\,r_{12}\lambda_{12},\lambda_3,r\lambda i\rangle = \sum_{r_{23}\lambda_{23}s}\langle\lambda_1(\lambda_2\lambda_3)\,r_{23}\lambda_{23},s\lambda|(\lambda_1\lambda_2)\,r_{12}\lambda_{12},\lambda_3,r\lambda\rangle\,|\lambda_1(\lambda_2\lambda_3)\,r_{23}\lambda_{23},s\lambda i\rangle. \quad (9.4)$$

Using these coefficients we can write the reduced kets of (9.2) in terms of (9.3) and use the orthogonality of the kets to equate appropriate terms, to give

$$\sum_{r_{12}\lambda_{12}i_{12}r}\langle r_{12}\lambda_{12}i_{12}|\lambda_1 i_1;\,\lambda_2 i_2\rangle\langle r\lambda i|\lambda_{12}i_{12};\,\lambda_3 i_3\rangle\langle\lambda_1(\lambda_2\lambda_3)\,r_{23}\lambda_{23},s\lambda|\,(\lambda_1\lambda_2)\,r_{12}\lambda_{23},\lambda_3,r\lambda\rangle$$
$$= \sum_{i_{23}}\langle r_{23}\lambda_{23}i_{23}|\lambda_2 i_2;\,\lambda i_3\rangle\langle s\lambda i|\lambda_1 i_1;\,\lambda_{23}i_{23}\rangle. \quad (9.5)$$

One can use the various unitary properties of the coupling coefficients to produce alternate forms of this relation. However, rather than investigate the properties of recoupling coefficients directly, we shall define a similar function – the 6-j symbol – and study it.

We define the 6-j symbol in terms of a sum over four 3-jm symbols

$$\begin{Bmatrix}\lambda_1\lambda_2\lambda_3\\ \mu_1\mu_2\mu_3\end{Bmatrix}_{r_1r_2r_3r_4} = (\lambda_1\mu_2\mu_3)_{r_1 i_1}{}^{j_2}{}_{j_3}\,(\mu_1\lambda_2\mu_3)_{r_2 j_1 i_2}{}^{j_3}\,(\mu_1\mu_2\lambda_3)_{r_3}{}^{j_1}{}_{j_2 i_3}\,(\lambda_1\lambda_2\lambda_3)_{r_4}{}^{i_1 i_2 i_3}. \quad (9.6)$$

(For those groups where raising multiplicity indices is not a trivial operation, the following property of the 6-j symbol would not be so simple.)

The complex conjugated 6-j symbol follows easily

$$\begin{Bmatrix}\lambda_1\lambda_2\lambda_3\\ \mu_1\mu_2\mu_3\end{Bmatrix}^*_{r_1r_2r_3r_4} = \begin{Bmatrix}\lambda_1^*\lambda_2^*\lambda_3^*\\ \mu_1^*\mu_2^*\mu_3^*\end{Bmatrix}_{r_1r_2r_3r_4}, \quad (9.7)$$

leading to the result that the 6-j symbol must be real for representations with real characters. Derome & Sharp investigated the properties of the 6-j in detail, giving in particular the symmetry properties, various recoupling properties, and the relation to group integrals of the form (5.12).

Permuting the columns of the 6-j symbol is simply performed by means of the appropriate 3-j interchange matrices, one for each of the four triples of (9.6); however, the phases do not necessarily cancel in the way they do for simply reducible groups. The symmetries may be generated by interchanging rows

$$\begin{Bmatrix}\lambda_1\lambda_2\lambda_3\\ \mu_1\mu_2\mu_3\end{Bmatrix}_{r_1r_2r_3r_4} = \begin{Bmatrix}\lambda_1^*\mu_2\mu_3^*\\ \mu_1^*\lambda_2\lambda_3^*\end{Bmatrix}_{r_4r_3r_2r_1}, \quad (9.8)$$

and interchanging columns

$$\begin{Bmatrix}\lambda_a\lambda_b\lambda_c\\ \mu_a^*\mu_b^*\mu_c^*\end{Bmatrix}_{s_a s_b s_c s_4} = \phi_{\mu_1}\phi_{\mu_2}\phi_{\mu_3}m(i,\lambda_1\mu_2^*\mu_3)_{s_1r_1}m(i,\mu_1\lambda_2\mu_3^*)_{s_2r_2}$$
$$\times\, m(i,\mu_1^*\mu_2\lambda_3)_{s_3r_3}m(i,\lambda_1\lambda_2\lambda_3)_{s_4r_4}\begin{Bmatrix}\lambda_1\lambda_2\lambda_3\\ \mu_1\mu_2\mu_3\end{Bmatrix}_{r_1r_2r_3r_4}. \quad (9.9)$$

Clearly cyclic permutations are of the same form, but without the complex conjugations on the left or the ϕ's on the right of (9.9).

The Racah back-coupling rule and the Biedenharn identity both generalize, the back-coupling rule being

$$\begin{Bmatrix}\lambda_1\lambda_2\lambda_3\\ \mu_1\mu_2\mu_3\end{Bmatrix}_{r_1r_2r_3r_4} = \sum_{\nu}|\nu|\,\phi_{\mu_2}\theta(\mu_1\lambda_2\mu_3^*r_2)\,\theta(\lambda_1\lambda_2\lambda_3r_4)$$
$$\times\,\theta(\lambda_1\mu_1\nu^*r)\begin{Bmatrix}\lambda_2\lambda_1\lambda_3\\ \mu_1\mu_2\nu\end{Bmatrix}_{r'rr_3r_4}\begin{Bmatrix}\lambda_1\mu_1\nu^*\\ \lambda_2\mu_2\mu_3\end{Bmatrix}_{r_1r_2r'r}, \quad (9.10)$$

where again we use (6.9) and (8.10) to simplify the expression.

The 6-j symbols are the elements of a unitary matrix

$$\sum_{\mu_3} |\lambda_3, u_3| \begin{Bmatrix} \lambda_1 \lambda_2 \lambda_3 \\ \mu_1 \mu_2 \mu_3 \end{Bmatrix}^*_{r_1 r_2 r_3 r_4} \begin{Bmatrix} \lambda_1 \lambda_2 \lambda_3' \\ \mu_1 \mu_2 \mu_3 \end{Bmatrix}_{r_1 r_2 r_3' r_4'} = \delta_{\lambda_3 \lambda_3'} \delta_{r_3 r_3'} \delta_{r_4 r_4'}. \tag{9.11}$$

Alternate forms of (9.11) are obtained by using the symmetries of the 6-j symbol.

We note also one recoupling relation among the 3-jm symbols not given by Derome and Sharp, since it is highly suggestive of a means of recursive calculation of the 3-jm symbols

$$(\lambda_1 \lambda_2 \lambda_3)_{r_4 i_1 i_2 i_3} \begin{Bmatrix} \lambda_1 \lambda_2 \lambda_3 \\ \mu_1 \mu_2 \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4} = (\lambda_1 \mu_2 \mu_3)_{r_1 i_1}{}^{j_2}{}_{j_3} (\mu_1 \lambda_2 \mu_3)_{r_2 j_1 i_2}{}^{j_3} (\mu_1 \mu_2 \lambda_3)_{r_3}{}^{j_1}{}_{j_2 i_3}. \tag{9.12}$$

The implied sums here are only in the multiplicity index on the left and $j_1 j_2$ and j_3 on the right making it a rather simple relation, since a suitably 'stretched' choice of the representations $\mu_1 \mu_2$ and μ_3 can reduce the sum to very few terms.

The recoupling coefficient of (9.4) can easily be written in terms of a 6-j symbol, by making a correspondence between (9.5) and Derome & Sharp's theorem 3. Four matrices K are required, as well as some interchange matrices.

$$\begin{aligned} &\langle (\lambda_1 \lambda_2) r_{12} \lambda_{12}, \lambda_3, r_2 \lambda | \lambda_1 (\lambda_2 \lambda_3) r_{23} \lambda_{23}, r_1 \lambda \rangle \\ &= |\lambda_{12}, \lambda_{23}|^{\frac{1}{2}} K(\lambda_1 \lambda_2 \lambda_{12})_{r_{12} s_{12}} K(\lambda_{12} \lambda_3 \lambda)_{r_2 s_2} K(\lambda_2 \lambda_3 \lambda_{23})^{s_{23} r_{23}} \\ &\quad \times K(\lambda_1 \lambda_{23} \lambda)^{s_1 r_1} \phi_{\lambda_2} m((12)\, \lambda_{12} \lambda_3 \lambda^*)_{s_2 t_2} m((132)\, \lambda_2 \lambda_3 \lambda_{23}^*)_{s_{23} t_{23}} \\ &\quad \times m((23)\, \lambda_1 \lambda_2 \lambda_{12}^*)_{s_{12} t_{12}} \begin{Bmatrix} \lambda_1 \lambda_{23} \lambda^* \\ \lambda_3^* \lambda_{12} \lambda_2 \end{Bmatrix}_{t_{12} t_{23} t_2 s_1}. \end{aligned} \tag{9.13}$$

After that expression, let us pause before continuing with any more definitions. We use the definition of the 6-j symbol to obtain a precise formula for all 6-j symbols with one scalar representation. From the definition we have

$$\begin{Bmatrix} \lambda_1 \lambda_2 \lambda_3 \\ \mu_1 \mu_2 1 \end{Bmatrix}_{11 r_3 r_4} = (\lambda_1 \mu_2 1)_{1 i_1}{}^{j_2}{}_1 (\mu_1 \lambda_2 1)_{1 j_1 i_2}{}^1 (\mu_1 \mu_2 \lambda_3)_{r_3}{}^{j_1}{}_{j_2 i_3} (\lambda_1 \lambda_2 \lambda_3)_{r_4}{}^{i_1 i_2 i_3}. \tag{9.14}$$

The indices j_2 may be lowered in the first 3-jm and raised in the third by appropriate 1-jm symbols which differ by ϕ_{μ_2}. The 1-jm $(1)^{11}$ is unity, thus

$$\begin{Bmatrix} \lambda_1 \lambda_2 \lambda_3 \\ \lambda_2^* \lambda_1 1 \end{Bmatrix}_{r_1 11 r_4} = (\lambda_2^* \lambda_1^* \lambda_3)_{r_3}{}^{j_1}{}_{i_3}{}^{j_2} (\lambda_1 \lambda_2 \lambda_3)_{r_4}{}^{i_1 i_2 i_3} \phi_{\lambda_1} (\lambda_2^* \lambda_2 1)_{1 j_1 i_2 1} (\lambda_1 \lambda_1^* 1)_{1 i_1 j_2 1}. \tag{9.15}$$

Noting the number of permutations to bring the latter two 3-jm's to their equivalent 1-jm's we have

$$\phi_{\lambda_1} (\lambda_2^* \lambda_2 1)_{1 j_1 i_2 1} (\lambda_1 \lambda_1^* 1)_{1 i_1 j_2 1} = |\lambda_1, \lambda_2|^{-\frac{1}{2}} (\lambda_2)_{j_1 i_2} (\lambda_1)_{i_1 j_2}. \tag{9.16}$$

Therefore
$$\begin{Bmatrix} \lambda_1 \lambda_2 \lambda_3 \\ \lambda_2^* \lambda_1 1 \end{Bmatrix}_{11 r_3 r_4} = |\lambda_1, \lambda_2|^{-\frac{1}{2}} (\lambda_2 \lambda_1 \lambda_3)_{r_3 i_2 i_1 i_3} (\lambda_1 \lambda_2 \lambda_3)_{r_4}{}^{i_1 i_2 i_3}. \tag{9.17}$$

The first 3-jm requires some reordering, and the sum over i_3 here cancels the factor $|\lambda_3|$ in the unitarity equation (3.3), giving

$$\begin{aligned} \begin{Bmatrix} \lambda_1 \lambda_2 \lambda_3 \\ \lambda_2^* \lambda_1 1 \end{Bmatrix}_{11 r_3 r_4} &= |\lambda_1, \lambda_2|^{-\frac{1}{2}} m((12)\, \lambda_1 \lambda_2 \lambda_3)_{r_3 r_4}, \\ &= |\lambda_1, \lambda_2|^{-\frac{1}{2}} \theta(\lambda_1 \lambda_2 \lambda_3 r_1)\, \delta_{r_3 r_4}. \end{aligned} \tag{9.18}$$

The interchange matrix takes care of the zero result if the 'triangular' condition is not satisfied.

10. The 9-j symbol

It is an obvious generalization of the previous section to define a recoupling coefficient for a direct product of four representations, λ_1, λ_2, μ_1 and μ_2

$$\langle(\lambda_1\lambda_2)\,r_1\lambda_3\,(\mu_1\mu_2)\,r_2\mu_3,\; s_3\nu_3|\,(\lambda_1\mu_1)\,s_1\nu_1\,(\lambda_2\mu_2)\,s_2\nu_2,\,r_3\nu_3\rangle. \tag{10.1}$$

We shall not discuss such quantities, although they reduce trivially to the recoupling coefficients of the previous section for the case of $\lambda_2 = 1$. Of more interest is the 9-j symbol defined by

$$\begin{Bmatrix}\lambda_1\lambda_2\lambda_3\\ \mu_1\mu_2\mu_3\\ \nu_1\nu_2\nu_3\end{Bmatrix}\begin{matrix}r_1\\ r_2\\ r_3\end{matrix}_{\;s_1s_2s_3} = \begin{Bmatrix}\lambda_1\lambda_2\lambda_3\\ \mu_1\mu_2\mu_3\\ \nu_1\nu_2\nu_3\end{Bmatrix}_{r_1r_2r_3s_1s_2s_3} = (\lambda_1\lambda_2\lambda_3)_{r_1\,i_1i_2i_3}(\mu_1\mu_2\mu_3)_{r_2\,j_1j_2j_3}(\nu_1\nu_2\nu_3)_{r_3\,k_1k_2k_3}$$
$$\times\,(\lambda_1\mu_1\nu_1)_{s_1}{}^{i_1j_1k_1}\,(\lambda_2\mu_2\nu_2)_{s_2}{}^{i_2j_2k_2}\,(\lambda_3\mu_3\nu_3)_{s_3}{}^{i_3j_3k_3}. \tag{10.2}$$

Of the two notations shown here for the 9-j symbol, we prefer the former (Vanagas 1971, p. 62) over the latter (Derome & Sharp 1965), since it places the multiplicity indices with their products. The latter is useful when the position of the multiplicity labels is significant. The representation labels in (10.1) were chosen to display its relation to (10.2), but it is left as an exercise for the reader to find the exact connexion between the two. It is to be noted, however, that the relation is much simpler than in the 3-ket case, as the various products occur in the same order in the two expressions. The 9-j symbol is much simpler than the 6-j by the further reason of the lack of the awkward raising and lowering of the indices in the definition. Indeed, the 1-jm symbol is not needed to define it. Consequently, the permutational symmetry of both rows and columns, the transposition about the diagonal, and the effect of complex conjugation, are all rather more obvious from the definition.

The 9-j symbol can be expressed as a sum of three 6-j symbols of the form (Derome 1965, p. 75)

$$\sum_{\kappa}|\kappa|\begin{Bmatrix}\lambda_1\mu_1\nu_1\\ \nu_2\nu_3\kappa\end{Bmatrix}_{t_2t_1r_3s_1}\begin{Bmatrix}\lambda_2\mu_2\nu_2\\ \mu_1\kappa\,\mu_3\end{Bmatrix}_{t_3r_2t_1s_2}\begin{Bmatrix}\lambda_3\mu_3\nu_3\\ \kappa\;\lambda_1\lambda_2\end{Bmatrix}_{r_1t_3t_2s_3}, \tag{10.3}$$

but many permutation matrices and other phases occur.

The 9-j symbol, with one of its entries the identity representation, reduces to a single 6-j:

$$\begin{Bmatrix}\lambda_1\lambda_2\mu\\ \lambda_3\lambda_4\mu^*\\ \nu\;\;\nu^*\;1\end{Bmatrix}\begin{matrix}r_1\\ r_2\\ 1\end{matrix}_{\;s_1s_21} = |\mu,\nu|^{-\frac{1}{2}}\phi_{\lambda_2}\phi_\nu\,\theta(\lambda_2\lambda_4\nu^*s_2)\,\theta(\lambda_3\lambda_4\mu^*r_2)\begin{Bmatrix}\lambda_1\lambda_3\;\nu\\ \lambda_4\lambda_2\mu\end{Bmatrix}_{r_1r_2s_2s_1}. \tag{10.4}$$

One could continue to define $3n$-j symbols for $n > 3$ but it is the 6-j and 9-j symbols that have been found to be most useful in physical applications

11. Transformations of basis

Before moving on to consider properties of a basis set of kets which has been chosen by means of some subgroup structure, it is worth considering the consequences of any change of basis within the representations.

Let the kets

$$|\lambda i_1\rangle,\,|\lambda i_2\rangle,\,\dots \tag{11.1}$$

form one such basis, and the kets

$$|\lambda x_1\rangle, |\lambda x_2\rangle, \ldots \tag{11.2}$$

form another, neither endowed with any particular properties. There will be a unitary matrix of dimension $|\lambda|$, which performs this change. We denote the (ix) element as $\langle \lambda i|\lambda x\rangle$; that is,

$$|\lambda x\rangle = \sum_i \langle \lambda i|\lambda x\rangle\, |\lambda i\rangle. \tag{11.3}$$

Kaplan (1962 *a*, *b*) studied the properties of such a transformation, in his work on the symmetric group. Moshinsky and his co-workers (see, for example, Moshinsky & Devi 1969) have used them for other groups, naming them 'transformation brackets'. We shall refer to them as transformation coefficients.

The usefulness of the transformation coefficients rests on three results – that there is a small number of them relative to the number of 3-*jm* symbols, they are quite easy to compute using projection techniques and given the 3-*jm* symbols for a mathematically convenient basis, one easily transforms to a physically useful basis. The reasoning for the last is as follows.

We rewrite the basic coupling equation (3.1) in terms of the new basis

$$|\lambda_1 i_1\rangle\, |\lambda_2 i_2\rangle = \sum_{r\lambda i} \langle r\lambda i|\lambda_1 i_1; \lambda_2 i_2\rangle |(\lambda_1\lambda_2)\, r\lambda i\rangle \tag{3.1}$$

$$= \sum_{x_1 x_2} \langle \lambda_1 x_1|\lambda_1 i_1\rangle\, \langle \lambda_2 x_2|\lambda_2 i_2\rangle |\lambda_1 x_1\rangle\, |\lambda_2 x_2\rangle$$

$$= \sum_{r\lambda i x} \langle r\lambda i|\lambda_1 i_1; \lambda_2 i_2\rangle\, \langle \lambda x|\lambda i\rangle |(\lambda_1\lambda_2) r\lambda x\rangle. \tag{11.4}$$

Re-expanding the product $|\lambda_1 x_1\rangle\, |\lambda_2 x_2\rangle$ in terms of the coupling coefficient for the new basis, and using the orthogonality of $|(\lambda_1\lambda_2)\, r\lambda x\rangle$ gives

$$\sum_{x_1 x_2} \langle \lambda_1 x_1|\lambda_1 i_1\rangle\, \langle \lambda_2 x_2|\lambda_2 i_2\rangle\, \langle r\lambda x|\lambda_1 x_1; \lambda_2 x_2\rangle = \sum_x \langle \lambda x|\lambda i\rangle\, \langle r\lambda i|\lambda_1 i_1; \lambda_2 i_2\rangle. \tag{11.5}$$

Note that we have used the same separation of the multiplicity, indeed one would have to make a special effort to change it, via (3.2). Using the definition and properties of 1-*jm* and 3-*jm* symbols gives also that

$$(\lambda_1\lambda_2\lambda_3)_{rx_1x_2x_3} = (\lambda_1\lambda_2\lambda_3)_{ri_1i_2i_3} \langle \lambda_1 i_1|\lambda_1 x_1\rangle\, \langle \lambda_2 i_2|\lambda_2 x_2\rangle\, \langle \lambda_3 i_3|\lambda_3 x_3\rangle. \tag{11.6}$$

The unitary properties of the transformation coefficients will ensure that all reference to the basis will disappear in the 3-*j* permutation matrices and the 6-*j* and 9-*j* symbols.

Kaplan's study of the transformation coefficients goes further than this. He studied them for the transformation from the Young–Yamanouchi basis for the symmetric group $(S_l \supset S_1 \oplus S_{l-1})$ to an arbitrary pair of symmetric subgroups $(S_l \supset S_a \oplus S_{l-a})$. He referred to them as recoupling coefficients, which can be seen to be somewhat of a misnomer. However, the representations occurring in such a basis are those given by the outer product of *S*-functions (the Littlewood–Richardson rule (Littlewood 1950, p. 94)), and this rule is also the one giving the Kronecker products for all unitary groups. One would thus expect these transformation coefficients between chains of symmetric groups to be related to the *n*-*j* symbols of U_N. Kramer has found this to be the case (Kramer 1968) and uses his results to explain the Regge symmetry of the 3-*jm* symbol of SU_2 (Kramer & Seligman 1969).

B. CHAINS OF GROUPS

In part A of this article, we have referred to the ith basis ket $|\lambda i\rangle$ of the (irreducible) ket representation λ of the group G. The index i has had no significance other than to enumerate the basis. However, where we have used the group R_3 as an example the index m has had a very special significance, namely it is the z-projection of the angular momentum j. This labelling is such that the basis kets are invariant under rotations about the z axis – that is, under the subgroup R_2. This gave rise to certain special results which we now generalize in the following manner.

12. Coupling (isoscalar) factors

Consider the space of the irreducible ket representation λ of dimension $|\lambda|$ of the group G. This is the space of kets where operations on any ket by any element R belonging to G gives another ket of the space. Let H be a subgroup of G. The restricted number of operators $R \in H$ will allow (in general) the representation space λ of G to be split into a sum of subspaces $\mu_1, \mu_2, \ldots$, each of which is closed under the action of H. For $G = R_3$ and $H = R_2$, the subspaces are all one-dimensional, and none occurs more than once, but in general the irreducible subspaces may occur several times and be of dimension larger than one. We write the reduction as

$$G \rightarrow H : \lambda \rightarrow \sum_{\mu} \alpha_{\mu} \mu$$

and refer to α_μ as the branching multiplicity for the representation μ, where α_μ has no simple connexion with the Kronecker product multiplicities, r, as can be seen from its dependence on the choice of the subgroup H. For example, for $G = R_3$ we could choose H to be the identity element of R_3, and then all the subspaces would be indistinguishable (and scalar) under this one operation of H, and the branching multiplicity would be $\alpha_0 = |j| = 2j+1$. When a multiplicity does occur, the subgroup cannot distinguish the properties of the appropriate subspaces and an *ad hoc* classification is required. We denote this by $a = 1, 2, \ldots, \alpha$.

Let us ignore the possible subgroup structures for H, and choose any basis $|\mu i\rangle$ for each of its representations. The collection of bases for all the ket representations μ of H contained in the ket representation λ of G will form a possible basis for λ, we denote a typical basis element by

$$|\lambda a \mu i\rangle. \tag{12.1}$$

Thus, where the summations were over the single index i before, they will now be over the collection $a\mu i$. On the other hand, since the kets $|\lambda a\mu i\rangle$ form a basis for ket representations of the two groups we have, (2.4),

$$O_R|\lambda a\mu i\rangle = \sum_{i'} |\lambda a\mu i'\rangle \mu(R)_{i'i} \quad \text{if} \quad R \in H, \tag{12.2}$$

and

$$O_R|\lambda a\mu i\rangle = \sum_{a'\mu' i'} |\lambda a'\mu' i'\rangle \lambda(R)_{a'\mu' i',\, a\mu i} \quad \text{if} \quad R \in G, \tag{12.3}$$

where (12.2) is a special case of (12.3) and thus

$$\lambda(R)_{a\mu i,\, a'\mu' i'} = \delta_{aa'} \delta_{\mu\mu'} \mu(R)_{ii'} \quad \text{if} \quad R \in H. \tag{12.4}$$

Likewise, from the definition of the coupling coefficients for both groups, we have

$$|\lambda_1 a_1 \mu_1 i_1\rangle\, |\lambda_2 a_2 \mu_2 i_2\rangle = \sum_{s\mu i} \langle s\mu i|\mu_1 i_1; \mu_2 i_2\rangle\, |(\lambda_1 a_1 \mu_1, \lambda_2 a_2 \mu_2)\, s\mu i\rangle, \tag{12.5}$$

and
$$|\lambda_1 a_1 \mu_1 i_1\rangle\,|\lambda_2 a_2 \mu_2 i_2\rangle = \sum_{r\lambda a\mu i} \langle r\lambda a\mu i|\lambda_1 a_1 \mu_1 i_1;\ \lambda_2 a_2 \mu_2 i_2\rangle|(\lambda_1\lambda_2)\,r\lambda a\mu i\rangle. \tag{12.6}$$

In the first equation (12.5) we have reduced the product ket with respect to H only, whereas in (12.6) the ket is reduced with respect to the larger group. The vectors

$$|(\lambda_1 a_1 \mu_1,\ \lambda_2 a_2 \mu_2)\,s\mu i\rangle \tag{12.7}$$

form a basis for the ket representation space μ of H. However, we must take linear combinations of these spaces to form spaces irreducible under G

$$|(\lambda_1 a_1 \mu_1, \lambda_2 a_2 \mu_2)\,s\mu i\rangle = \sum_{r\lambda a} \langle r\lambda a\mu i|\,(\lambda_1 a_1 \mu_1,\ \lambda_2 a_2 \mu_2)\,s\mu i\rangle|(\lambda_1\lambda_2)\,r\lambda a\mu i\rangle. \tag{12.8}$$

Since this transformation is, from the point of view of H, of the form (3.2), the coefficients must be independent of i. Therefore, we rewrite (12.8) as

$$|(\lambda_1 a_1 \mu_1,\ \lambda_2 a_2 \mu_2)\,s\mu i\rangle = \sum_{r\lambda a} \langle r\lambda a\mu s|\lambda_1 a_1 \mu_1;\ \lambda_2 a_2 \mu_2\rangle|(\lambda_1\lambda_2)\,r\lambda a\mu i\rangle. \tag{12.9}$$

The unitary properties of these 'isoscalar factors' follow immediately

$$\sum_{r\lambda a} \langle r\lambda a\mu s|\,\lambda_1 a_1 \mu_1;\lambda_2 a_2 \mu_2\rangle\,\langle \lambda_1 a_1' \mu_1';\ \lambda_2 a_2' \mu_2'|r\lambda a\mu s'\rangle = \delta_{a_1 a_1'}\delta_{a_2 a_2'}\delta_{\mu_1 \mu_1'}\delta_{\mu_2' \mu_2}\delta_{ss'}, \tag{12.10}$$

$$\sum_{s a_1 \mu_1 a_2 \mu_2} \langle r\lambda a\mu s|\,\lambda_1 a_1 \mu_1;\lambda_2 a_2 \mu_2\rangle\,\langle \lambda_1 a_1 \mu_1;\lambda_2 a_2 \mu_2\,|r'\lambda' a'\mu s\rangle = \delta_{rr'}\,\delta_{\lambda\lambda'}\,\delta_{aa'}. \tag{12.11}$$

We can use the reduction (12.9) to complete the reduction of (12.5), which is then to be compared with (12.6),

$$\langle r\lambda a\mu i|\,\lambda_1 a_1 \mu_1 i_1;\lambda_2 a_2 \mu_2 i_2\rangle = \sum_s \langle r\lambda a\mu s|\lambda_1 a_1 \mu_1;\ \lambda_2 a_2 \mu_2\rangle\,\langle s\mu i|\mu_1 i_1;\ \mu_2 i_2\rangle. \tag{12.12}$$

This is Racah's result (1949) for the factorization of coupling coefficients defined for a group chain. It must rate as one of the most important results in the theory of coupling coefficients. (Consistency in terminology demands that isoscalar factors be called coupling factors.)

It is instructive to again look at the group R_3. Using representation theory only, we see that for the group chain $R_3 \supset R_2$, all multiplicity labels disappear, and also that the coupling coefficient for the subgroup R_2, is at most a phase factor. Thus, these isoscalar factors reduce to the usual Clebsch–Gordan coefficients, but where we have the additional restriction that $m = m_1 + m_2$. This follows from the character theory of R_2, and is thus a consequence of the subgroup chosen for the basis, and not a consequence of the coupling coefficients for R_3.

13. 1-*jm* and 3-*jm* factors

The Racah factorization lemma just derived suggests a similar factorization of the 1-*jm* and 3-*jm* symbols, but first a few words on the notation.

The tensorial notation of part A has largely outlived its usefulness, and we return to the usual notation. When we wish to show the subgroup structure, we write the labels corresponding to the different groups on different lines

$$(\lambda_1\lambda_2\lambda_3)_{ra_1\mu_1 i_1\ a_2\mu_2 i_2\ a_3\mu_3 i_3} \equiv \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^r. \tag{13.1}$$

We write the expression equivalent to Racah's factorization (12.12) as

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^r = \sum_s \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3 \end{pmatrix}^r_s \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^s. \tag{13.2}$$

This 3-*jm* factor will then have the unitary properties

$$\sum_{r\lambda_3 a_3} \frac{|\lambda_3|}{|\mu_3|} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3 \end{pmatrix}^{*r}_s \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1'\mu_1' & a_2'\mu_2' & a_3\mu_3 \end{pmatrix}^r_{s'} = \delta_{a_1 a'}\delta_{a_2 a_2'}\delta_{\mu_1\mu_1'}\delta_{\mu_2\mu_2'}\delta_{ss'}, \tag{13.3}$$

$$\sum_{a_1\mu_1 a_2\mu_2 s} \frac{|\lambda_3|}{|\mu_3|} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3 \end{pmatrix}^{*r}_s \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3' \\ a_1\mu_1 & a_2\mu_2 & a_3'\mu_3 \end{pmatrix}^{r'}_s = \delta_{rr'}\delta_{a_3 a_3'}\delta_{\lambda_3\lambda_3'}. \tag{13.4}$$

Let us now, as an exercise, use the definitions of the 3-*jm* symbols (5.1), the coupling (isoscalar) factor property (12.12) and the definition (13.2) to calculate the correspondence between the coupling and 3-*jm* factors. In so doing we shall produce a 1-*jm* factor.

For the group G we have

$$\langle r\lambda a\mu i|\lambda_1 a_1\mu_1 i_1;\ \lambda_2 a_2\mu_2 i_2\rangle = \sum_{r_3 a_3\mu_3 i_3} |\lambda|^{-\frac{1}{2}} K(\lambda_1\lambda_2\lambda)_{rr_3}(\lambda)^{a\mu i,\, a_3\mu_3 i_3} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3^* \\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^{r_3}. \tag{13.5}$$

Factorizing both the coupling coefficient and the 3-*jm* symbol gives

$$\begin{aligned}\sum_s \langle r\lambda a\mu s|\lambda_1 a_1\mu_1;\ \lambda_2 a_2\mu_2\rangle\langle s\mu i|\mu_1 i_1;\ \mu_2 i_2\rangle \\ = \sum_{r_3 s_3 a_3\mu_3 i_3} |\lambda|^{\frac{1}{2}} K(\lambda_1\lambda_2\lambda)_{rr_3}(\lambda)^{a\mu i,\, a_3\mu_3 i_3} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda^* \\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3 \end{pmatrix}^{r_3}_{s_3} \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ i_1 & i_2 & i_3 \end{pmatrix}^{s_3}, \quad (13.6) \\ = \sum_{ss_4 i_4} \langle r\lambda a\mu s|\lambda_1 a_1\mu_1;\ \lambda_2 a_2\mu_2\rangle |\mu|^{\frac{1}{2}} K(\mu_1\mu_2\mu)_{ss_4}(\mu)^{ii_4} \begin{pmatrix} \mu_1 & \mu_2 & \mu^* \\ i_1 & i_2 & i_4 \end{pmatrix}^{s_4}, \quad (13.7)\end{aligned}$$

where the second equality comes from the definition of the 3-*jm* symbol for the subgroup H. By using the unitarity (5.5) of this same symbol many of the sums are removed, namely

$$\begin{aligned}\sum_s \langle r\lambda a\mu s|\lambda_1 a_1\mu_1;\ \lambda_2 a_2\mu_2\rangle |\mu|^{\frac{1}{2}} K(\mu_1\mu_2\mu)_{ss_4}(\mu)^{ii_4} \\ = \sum_{r_3 a_3\mu_3 i_3 s_3} |\lambda|^{\frac{1}{2}} K(\lambda_1\lambda_2\lambda)_{rr_3}(\lambda)^{a\mu i,\, a_3\mu_3 i_3} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda^* \\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3 \end{pmatrix}^{r_3}_{s_3} \delta_{s_3 s_4}\delta_{\mu_3\mu^*}. \quad (13.8)\end{aligned}$$

The use of the unitarity of $K(\mu_1\mu_2\mu)$ and the 1-*jm* symbol (μ) leads, after a change in notation, to

$$\langle r\lambda a\mu s|\lambda_1 a_1\mu_1;\ \lambda_2 a_2\mu_2\rangle = \sum_{r's'a'} |\lambda|^{\frac{1}{2}} |\mu|^{-\frac{1}{2}} K(\lambda_1\lambda_2\lambda)_{rr'} K(\mu_1\mu_2\mu)^{ss'}(\lambda)^{a\mu,\, a'\mu^*} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda^* \\ a_1\mu_1 & a_2\mu_2 & a'\mu^* \end{pmatrix}^{r'}_{s'}, \tag{13.9}$$

where the factor

$$(\lambda)^{a\mu,\, a'\mu^*} = (\lambda)^{a\mu i,\, a'\mu' i'}(\mu)_{ii'} \tag{13.10}$$

is clearly a 1-*jm* factor that could have been defined either as a special case of the 3-*jm* factor or by an expression analogous to (13.2). This calculation sheds light on the 1-*jm* factor, namely that it is of the form $\delta_{\mu'\mu^*}$. This can be traced to the isomorphism between the representation μ and its complex conjugate – for all groups. (For R_2, since $m^* = -m$, we have immediately that $(j)_{mm'} = \pm\delta_{m'-m}$ where only the phase is to be found.)

Assuming a 'sensible' choice of the separation of the product multiplicities in the coupling coefficients for both groups (see equations (5.2)), (13.9) simplifies enormously, namely

$$\langle r\lambda a\mu s|\lambda_1 a_1\mu_1;\ \lambda_2 a_2\mu_2\rangle_{\text{sensible}} = |\lambda|^{\frac{1}{2}}\,|\mu|^{-\frac{1}{2}}\sum_{a'}(\lambda)^{a\mu,\,a'\mu^*}\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda^*\\ a_1\mu_1 & a_2\mu_2 & a'\mu^*\end{pmatrix}^r_s. \tag{13.11}$$

The symmetries of the 3-*jm* factor follow immediately from those of the 3-*jm* symbols; for example,

$$\sum_{r's'} m(\pi,\lambda_1\lambda_2\lambda_3)_{rr'}\,m(\pi^{-1},\mu_1\mu_2\mu_3)_{ss'}\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ a_1\mu_1 & a_2\mu_2 & a_3\mu_3\end{pmatrix}^{r'}_{s'} = \begin{pmatrix}\lambda_a & \lambda_b & \lambda_c\\ a_a\mu_a & a_b\mu_b & a_c\mu_c\end{pmatrix}^r_s, \tag{13.12}$$

where (abc) is the permutation π of (123). Note the occurrence of the inverse transformation in the subgroup. This will only be of consequence for cyclic permutations of non-simple phase representations.

Equation (13.12) when used in (13.9) gives a much more general statement than Racah (1949) of the 'reciprocity' of coupling factors. His equations ((46)–(49)) are valid only for multiplicity free products of representations with real characters. Assuming the 'sensible' phase choice of (5.2) we have, for example,

$$\langle\lambda_2 a_2\mu_2;\ \lambda_1 a_1\mu_1|r\lambda a\mu s\rangle = \theta(\lambda_1\lambda_2\lambda_3 r)\,\theta(\mu_1\mu_2\mu_3 s)\,\langle\lambda_1 a_1\mu_1;\ \lambda_2 a_2\mu_2|r\lambda a\mu s\rangle. \tag{13.13}$$

It is suggested that the reader obtain the expression for the (13) permutation by substituting the inverse of (13.9) into (13.12).

The many relations of §§ 9 and 10 between 6-*j* and 9-*j* symbols and 1-*jm* and 3-*jm* symbols, can easily be written in terms of 1-*jm* and 3-*jm* factors. The 6-*j* and 9-*j* symbols being independent of the basis do not factorize, but the factors can be used to relate the symbols of a group to those of a subgroup, for example, (9.12) becomes

$$\sum_{r_4}\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ a_1\sigma_1 & a_2\sigma_2 & a_3\sigma_3\end{pmatrix}^{r_4}_{s_4}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\end{Bmatrix}_{r_1r_2r_3r_4}$$

$$= \sum_{\substack{s_1s_2s_3b_1b_2b_3\\ b_1'b_2'b_3'\rho_1\rho_2\rho_3}}(\mu_1)^{b_1\rho_1,\,b_1'\rho_1^*}(\mu_2)^{b_2\rho_2,\,b_2'\rho_2^*}(\mu_3)^{b_3\rho_3,\,b_3'\rho_3^*}\begin{pmatrix}\lambda_1 & \mu_2^* & \mu_3\\ a_1\sigma_1 & b_2'\rho_2^* & b_3\rho_3\end{pmatrix}^{r_1}_{s_1}$$

$$\times\begin{pmatrix}\mu_1 & \lambda_2 & \mu_3^*\\ b_1\rho_1 & a_2\sigma_2 & b_3'\rho_3^*\end{pmatrix}^{r_2}_{s_2}\begin{pmatrix}\mu_1^* & \mu_2 & \lambda_3\\ b_1'\rho_1^* & b_2\rho_2 & a_3\sigma_3\end{pmatrix}^{r_3}_{s_3}\begin{Bmatrix}\sigma_1 & \sigma_2 & \sigma_3\\ \rho_1 & \rho_2 & \rho_3\end{Bmatrix}_{s_1s_2s_3s_4}. \tag{13.14}$$

The notation has been changed slightly here, λ and μ denoting representations of G, and σ and ρ of H.

14. Canonical chains

The presence of all the labels in the above two sections used to account for the possibility of branching multiplicities in the imbedding of H in G, gave a rather awesome aspect to the equations. The equations simplify considerably when no such multiplicities occur, and we shall loosely refer to this case as a canonical imbedding.

Baird, Biedenharn, Louck, Moshinsky and their co-workers have done much analytic work on the properties of these coefficients for the canonical chain of the unitary groups $U_n \supset U_{n-1} \supset \ldots$ (for references see Louck 1970, Moshinsky & Devi 1969). Sharp (1970) and Wong (1971) have used their techniques to look at the cases $R_n \supset R_{n-1}$ and $Sp_n \supset Sp_{n-2}$ (the second is not canonical). We restrict ourselves to a few comments on the 1-*jm* factor.

The 1-*jm factor*

In the previous section we remarked that the 1-*jm* factor took on a particularly simple form for the canonical case

$$(\lambda)_{\mu\mu'} = \theta(\lambda\mu)\,\delta_{\mu'\mu^*}. \tag{14.1}$$

From the permutation symmetry we obtain

$$\theta(\lambda^*\mu^*) = \phi_\lambda\phi_\mu\theta(\lambda\mu). \tag{14.2}$$

If both λ and μ are real, then clearly $\phi_\lambda = \phi_\mu$ or λ does not contain μ. Baird & Biedenharn (1965) have investigated the 1-*jm* symbols for unitary group chains, and it is a trivial matter to factorize their result

$$\text{for}\quad SU_n \supset U_{n-1}\,\theta(\lambda\mu) = (-)^{m_{\max}-m}, \tag{14.3}$$

where μ is a partition of m and $\mu_{\max}$ is the representation of greatest weight contained in λ. This expression must be compatible with (14.2) for real representations, but it is easy to show that it is not compatible with the choice of ϕ_λ wanted in § 8, equation (8.10). As Baird & Biedenharn note, there are phase choices to make in obtaining (14.3) and we would recommend that the choice was made to make the 1-*jm* symbols compatible with (8.10). Baird & Biedenharm obtain their phase by requiring maximal states positive.

C. TENSOR OPERATORS

The previous parts have defined and discussed the ket representations. They used the idea of the separation of the space into the irreducible subspaces, together with elementary group and vector space theory, to define coupling and recoupling (Racah) coefficients and *jm* and *j* symbols. (These are all related to the elements of the matrices which perform various changes of basis in the Hilbert space.) By keeping the approach rather abstract we were able to derive several new properties of the coefficients, and to put known properties in a more general framework.

Although the above ideas of 'coupling' kets have certain direct applications in physics – the coefficients for the three dimensional rotation group, R_3, perform the 'addition' of angular momenta – it is clear that the obtaining of the matrix elements of operators is the central problem in physics. For example, one must know the matrix elements of the energy and the transition operators. It is the purpose of this part to study such matrix elements. The key theorem will be proved in § 16. This is the Wigner–Eckart theorem. It states that the matrix element of any operator between any pair of vectors, is proportional to a 3-*jm* symbol (or coupling coefficient). This is if we choose the suitable group theoretic basis. We chose the basis for the ket space in § 2, and shall do this for the operator space in § 15. But first we must discuss one or two basic properties of the space of linear operators, in particular its reducibility under the group, in order to lead up to the definition of an irreducible tensor–operator representation space.

In this part we shall extend the terminology 'ket representation' to 'tensor representation', although the group elements are not (and cannot be) mapped homomorphically to the kets and tensors. Instead the kets and tensors have well defined – standard – transformation properties under the group. Their transformation properties are such that they give rise to irreducible matrix representations. Other authors have used the terms irreducible tensorial set (Fano & Racah 1959) or basis for an irreducible representation (Moshinsky 1963).

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Owing to the existence of the matrix $K(\lambda_1\lambda_2\lambda_3)$ linking the coupling coefficient and the 3-*jm* symbol, in this part we shall avoid the use of Wigner coupling (Clebsch–Gordan) coefficients by doing all our coupling with *jm* symbols. This is equivalent to insisting on the sensible choice of phase in (5.2), and hence no generality is lost.

In § 16 we shall discuss the Wigner–Eckart theorem and the definition of an arbitrary tensor operator. Section 17 is devoted to a discussion of some of the problems associated with choosing linear combinations of tensors to produce unitary operators. A special and important class of unitary operators are the operators O_R associated with the action of the abstract group on the Hilbert space. In § 18 we use these ideas to derive relations between the 3-*jm* symbols and the representation matrices.

Judd (1963) and Vanagas (1971) in their discussions on atomic and nuclear structure calculations find that they often are required to evaluate matrix elements of products of tensors. Section 19 produces some generalizations of formulae obtained by these authors for R_3 and S_n.

In part B we considered some of the consequences of choosing the basis kets using the action of a subgroup. In § 20 we again consider the consequences of such a choice. The properties of arbitrary tensors are not quite as simple as one might expect, especially with regard to unitary linear combinations. On the other hand, such considerations lead to some powerful results in terms of the group operators.

15. Tensor representations

Given an N-dimensional linear vector space, it is well known (Messiah 1965, p. 62) how to define a set of linear operators. The full set of linear operators forms an N^2-dimensional linear space. The positive-definite scalar product of a Hilbert space allows an identification of the space with its dual

$$(|a\rangle)^\dagger = \langle a|,$$

and the construction of an orthonormal (in addition to a linearly independent) basis

$$\langle i|j\rangle = \delta_{ij} \quad (i,j = 1, 2, \ldots, N).$$

These concepts were used in part A, but we may also immediately choose a complete, linearly independent basis for the operator space, a typical element being written

$$|i\rangle\langle j|.$$

We refer the readers completely unfamiliar with the concept and properties of the operator basis so defined, to Messiah (1965), p. 250.

Following the procedure of part A, we use the action of a finite or compact (but otherwise arbitrary) group G, to find a basis $|x\lambda i\rangle$ for the Hilbert space. x serves to enumerate distinct subspaces with the same transformation properties given by the representation label λ and i is an arbitrary choice of basis within these subspaces (see § 2). The action of $R \in G$ is given by

$$O_R|x\lambda i\rangle = \sum_j |x\lambda j\rangle \lambda(R)_{ji}. \tag{2.4}$$

It follows that one may choose as a basis for the linear operators, the operators

$$|x_1\lambda_1 i_1\rangle\langle x_2\lambda_2 i_2|. \tag{15.1}$$

The action of the group on these operators is given by

$$O_R|x_1\lambda_1 i_1\rangle\langle x_2\lambda_2 i_2|\,O_{R^{-1}} = O_R|x_1\lambda_1 i_1\rangle\,(O^{\dagger}_{R^{-1}}|x_2\lambda_2 i_2\rangle)^{\dagger}$$
$$= \sum_{j_1 j_2}|x_1\lambda_1 j_1\rangle\langle x_2\lambda_2 j_2|\,\lambda_1(R)_{j_1 i_1}\lambda_2(R)^{j_2 i_2}. \qquad (15.2)$$

The matrix $\lambda_2(R)^{j_2 i_2}$ is similar to the matrix $\lambda_2^*(R)_{j_3 i_3}$ and may be transformed into it by using the 1-*jm* symbol (4.4). Thus the basis operators (15.1) transform as the Kronecker product of the two representations λ_1 and λ_2^*.

We now take linear combinations of the above operators in order to obtain operators that transform as an irreducible representation. The combinations

$$(r\lambda i(x_1\lambda_1, x_2\lambda_2))_U = |\lambda|^{\frac{1}{2}}\,U_{rs}(\lambda_1\lambda\lambda_2)_s{}^{i_1}{}_{ii_2}|x_1\lambda_1 i_1\rangle\langle x_2\lambda_2 i_2| \qquad (15.3)$$

with U any non-singular matrix independent of $i_1 i_2$ and i, will form a linearly independent set which will also span the space.

The transformation property of the operators of (15.3) follows from (15.2), (4.4) and (5.7).

$$O_R(\lambda i(x_1\lambda_1, x_2\lambda_2))_U\,O_{R^{-1}} = (r\lambda j(x_1\lambda_1, x_2\lambda_2))_U\,\lambda(R)_{ji}. \qquad (15.4)$$

By setting the matrix U of (15.3) to the unit matrix, the definition reduces to that of Judd (1963, pp. 71, 101) for R_3. One is able to use as U the various matrices of part A to effect changes in the order of $\lambda_1\lambda_2$ and λ in this definition or to use the coupling coefficient (5.1). We shall take $U_{rs} = \delta_{rs}$ as our definition for our basis for tensor operators.

Most authors (see, for example, Messiah 1965, p. 572) refer to the set of operators $(r\lambda i(x_1\lambda_1, x_2\lambda_2))$ for all i, as the tensor, referring to each operator as a component. However, we prefer to use the term tensor for each operator in this irreducible basis (with respect to the group G). The set for all i is referred to as basis for a space of tensors transforming as λ or simply as a tensor representation. This parallels the terms used for the representation kets.

Thus a standard basis tensor is defined by

$$(r\lambda i(x_1\lambda_1, x_2\lambda_2)) = |\lambda|^{\frac{1}{2}}(\lambda_1\lambda\lambda_2)_r{}^{i_1}{}_{ii_2}|x_1\lambda_1 i_1\rangle\langle x_2\lambda_2 i_2|. \qquad (15.5)$$

If the dimension of the Hilbert space is N, then we have N^2 linearly independent tensor operators.

16. The Wigner–Eckart theorem

The matrix element of the basis tensors (15.5) between arbitrary (basis) bras and kets follows trivially

$$\langle x_1'\lambda_1' i_1|\,(r\lambda i(x_1\lambda_1, x_2\lambda_2))\,|x_2'\lambda_2' i_2\rangle = |\lambda|^{\frac{1}{2}}\,(\lambda_1\lambda\lambda_2)_r{}^{i_1}{}_{ii_2}\,\delta_{x_1x_1'}\delta_{\lambda_1\lambda_1'}\delta_{x_2x_2'}\delta_{\lambda_2\lambda_2'}. \qquad (16.1)$$

We now define, as is usual (Stone 1961; Messiah 1965, p. 1094), a tensor operator as any linear combination of the basis operators that retains the transformation property (15.4). Thus Q_i^{λ} is said to be a tensor operator (member of the tensorial set $\boldsymbol{Q}^{\lambda}$) if

$$O_R\,Q_i^{\lambda}\,O_{R^{-1}} = Q_j^{\lambda}\,\lambda(R)_{ji}. \qquad 16.2)$$

We reverse the argument that led to (15.4) to obtain the resolution of Q_i^{λ} in terms of the basis tensors. We have

$$Q_i^{\lambda} = \sum_{rx_1\lambda_1x_2\lambda_2} q(r\lambda x_1\lambda_1 x_2\lambda_2)\,(r\lambda i(x_1\lambda_1, x_2\lambda_2)), \qquad (16.3)$$

where the coefficient q depends on all the labels specified. The matrix elements of this generalized tensor are thus given by

$$\langle x_1\lambda_1 i_1|\, Q_i^{\lambda}\, |x_2\lambda_2 i_2\rangle = (\lambda_1\lambda\lambda_2)_r{}^{i_1}{}_{ii_2}\langle x_1\lambda_1\|\, Q^{\lambda r}\,\|x_2\lambda_2\rangle, \tag{16.4}$$

where the implied sum is only over r. The number $\langle x_1\lambda_1\|\, Q^{kr}\,\|x_2\lambda_2\rangle$ is known as the reduced matrix element and is given by

$$\langle x_1\lambda_1\|\, Q^{\lambda r}\,\|x_2\lambda_2\rangle = |\lambda|^{\frac{1}{2}} q(r\lambda x_1\lambda_1 x_2\lambda_2). \tag{16.5}$$

These two equations are a direct consequence of the fact that the operators of (15.5) are a complete basis for operators acting on the Hilbert space.

Equation (16.4) is known as the Wigner–Eckart theorem, and is precisely the result given by Messiah (1965, p. 1094). In terms of this language the reduced matrix element of a basis tensor is given as

$$\langle x_1\lambda_1\|\,(r'\lambda(x_1'\lambda_1', x_2'\lambda_2'))\,r\,\|x_2\lambda_2\rangle = |\lambda|^{\frac{1}{2}}\,\delta_{rr'}\,\delta_{x_1x_1'}\,\delta_{\lambda_1\lambda_1'}\,\delta_{x_2x_2'}\,\delta_{\lambda_2\lambda_2'}. \tag{16.6}$$

17. Unitary operators

For many purposes it is useful to have unitary operators. We can make the distinction between operators which are unitary over the whole Hilbert space, and those which are unitary over some specified subspace. The basis operators of (15.1) are clearly not unitary in either sense, except for those of the form

$$|x\lambda i\rangle\,\langle x\lambda i|, \tag{17.1}$$

which are unitary over that one dimensional subspace spanned by the ket $|x\lambda i\rangle$.

Some of the linear combinations of (15.5) will be unitary of a larger subspace. We have (see Messiah 1965, p. 254, for adjoint operators on a Hilbert space)

$$(r\lambda i(x_1\lambda_1, x_2\lambda_2))^{\dagger} = |\lambda|^{\frac{1}{2}}\phi_{\lambda_1}(\lambda_1\lambda\lambda_2)_r{}^{i_1}{}_i{}^{i_2}\,|x_2\lambda_2 i_2\rangle\,\langle x_1\lambda_1 i_1|. \tag{17.2}$$

We used the property of the complex conjugated 3-jm symbol

$$\phi_{\lambda_1}(\lambda_1\lambda\lambda_2)^s{}_{i_1}{}^{ii_2} = (\lambda_1)_{i_1 j}(\lambda_1^*\lambda\lambda_2)^{sjii_2} = \{(\lambda_1\lambda\lambda_2)_s{}^{i_1}{}_{ii_2}\}^*.$$

Thus (not summing over i or r)

$$\begin{aligned}(r\lambda i(x_1\lambda_1, x_2\lambda_2))^{\dagger}\,(r\lambda i(x_1\lambda_1, x_2\lambda_2)) &= |\lambda|\,\phi_{\lambda_1}(\lambda_1\lambda\lambda_2)_r{}^{i_1}{}_{ii_2}(\lambda_1\lambda\lambda_2)^r{}_{i_1'}{}^{ii_2}\,\delta_{x_1x_2}\delta_{\lambda_1\lambda_2}|x_1\lambda_1 i_1\rangle\,\langle x_1\lambda_1 i_1| \\ &= \delta_{x_1x_2}\delta_{\lambda_1\lambda_2}\sum_{i_1}|x_1\lambda_1 i_1\rangle\,\langle x_1\lambda_1 i_1|.\end{aligned} \tag{17.3}$$

The term

$$\sum_{i_1}|x_1\lambda_1 i_1\rangle\,\langle x_1\lambda_1 i_1| \tag{17.4}$$

is the unit operator of the ket representation space $x_1\lambda_1$ and hence all operators (all $r\lambda$ which occur in the Kronecker product $\lambda_1\times\lambda_1^*$)

$$(r\lambda i(x_1\lambda_1, x_1\lambda_1)) \tag{17.5}$$

are unitary operators over the representation space $x_1\lambda_1$. It was to ensure this that the normalization $|\lambda|^{\frac{1}{2}}$ was included in the definition (15.6).

It is clearly impossible to simply add operators of the form (17.5) to obtain operators which are unitary of the whole space – with the exception of the scalar (identity) operator. This is because a given symmetry type $r\lambda$ will not always occur in every product $\lambda_1\times\lambda_1^*$ (excepting $r = 1$, $\lambda = 1$).

We note that as also in many equations of the previous parts, the right-hand side of (17.3) includes the condition that the appropriate triple Kronecker product contains the identity.

18. The group operators as tensors

Consider the operator O_R associated with the abstract group element R. Because O_R acts within each ket representation, and is a unitary operator, it follows that it is a linear combination of the operators (17.5)

$$O_R = \sum_{\lambda_1 \lambda} U(R, r\lambda_1 \lambda i) \sum_{x_1} (r\lambda i(x_1 \lambda_1, x_1 \lambda_1)). \tag{18.1}$$

The independence of U from x_1 follows from (2.4). (Note there is an implied sum over r and i.) Comparing the action of both sides of this equation on an arbitrary ket, gives

$$\lambda(R)_{ji} = \sum_{\lambda'} U(R, r\lambda\lambda' i') \, |\lambda'|^{\frac{1}{2}} (\lambda\lambda'\lambda)_{r\,\,i'i}^{\,j}, \tag{18.2}$$

or, by using the unitarity of the 3-*jm*

$$U(R, r\lambda\lambda' i') = |\lambda'|^{\frac{1}{2}} \phi_\lambda(\lambda\lambda'\lambda)^{r}{}_{j}{}^{i'i} \lambda(R)_{ji}. \tag{18.3}$$

Alternatively, multiplying (18.2) by its complex conjugate and summing over i and j gives

$$\sum_{r\lambda' i'} U(R_1 r\lambda\lambda' i') \, U(R_1 r\lambda\lambda' i')^* = |\lambda|. \tag{18.4}$$

Consider the transformation property of both sides of (18.1)

$$O_S O_R O_{S^{-1}} = \sum_{\lambda_1 \lambda} U(R, r\lambda_1 \lambda i) \, O_S \sum_{x} (r\lambda i(x\lambda_1, x\lambda_1)) \, O_{S^{-1}}, \tag{18.5}$$

thus

$$O_{SRS^{-1}} = \sum_{\lambda_1 \lambda} U(R, r\lambda_1 \lambda i) \, \lambda(S)_{ji} \sum_{x} (r\lambda j(x\lambda_1, x\lambda_1)), \tag{18.6}$$

and

$$U(SRS^{-1}, r\lambda_1 \lambda j) = U(R, r\lambda_1 \lambda i) \, \lambda(S)_{ji}. \tag{18.7}$$

This is a statement concerning the relations between the different elements belonging to the same class of the group.

By studying the group algebra, one is able to make definite statements concerning this problem. For example, class sums are invariant under the group and thus we must have for all R and λ

$$|U(R, 1\lambda 11)| > 0. \tag{18.8}$$

We shall return to the study of group operators in § 20.

19. Coupling tensors

It is clear that the product of any two tensors P_i^λ and Q_j^μ may be reduced to a sum of irreducible tensors by using the 3-*jm* symbols. Various formulae may be derived to relate the reduced matrix element of the product to the reduced matrix elements of P and Q. Judd (1963, p. 71) carries out the analysis for R_3 to later use the results extensively in atomic structure calculations. Vanagas (1971, p. 61) does likewise for S_n using his results for nuclear computations. Our results reduce to the above, except that Judd uses the Condon and Shortley phases (Clebsch–Gordan coefficients) for coupling both kets and tensors. His results thus reduce to ours only if one inserts the matrix K linking definitions (5.1) and (5.2). Vanagas uses our phases but different normalizations.

Thus we define coupled kets by

$$|(x_1\lambda_1 x_2\lambda_2)\, r\lambda i\rangle = |\lambda|^{\frac{1}{2}}\phi_\lambda(\lambda_1\lambda_2\lambda)^{ri_1i_2}{}_i|x_1\lambda_1 i_1\rangle\,|x_2\lambda_2 i_2\rangle, \tag{19.1}$$

and tensors by

$$\{P^{\lambda_1}Q^{\lambda_2}\}^{r\lambda}_i = |\lambda|^{\frac{1}{2}}\phi_\lambda(\lambda_1\lambda_2\lambda)^{ri_1i_2}{}_i P^{\lambda_1}_{i_1}Q^{\lambda_2}_{i_2}. \tag{19.2}$$

We now evaluate the coupled tensor's reduced matrix element in terms of the reduced matrix elements of the uncoupled tensor. We do this, both for the general case, and for a very important special case.

In the general case we assume no information concerning the reduced matrix elements of P and Q. We have for the general matrix element (using (16.4) and (19.2))

$$\begin{aligned}\langle x_1\lambda_1 i_1|\,\{P^{\kappa_1}Q^{\kappa_2}\}^{r\kappa}_k\,|x_2\lambda_2 i_2\rangle &= (\lambda_1\kappa\lambda_2)_s{}^{i_1}{}_{ki_2}\langle x_1\lambda_1\|\,\{P^{\kappa_1}Q^{\kappa_2}\}^{r\kappa s}\,\|x_2\lambda_2\rangle \\ &= |\kappa|^{\frac{1}{2}}\phi_\kappa(\kappa_1\kappa_2\kappa)^{rk_1k_2}{}_k\langle x_1\lambda_1 i_1|\,P^{\kappa_1}_{k_1}Q^{\kappa_2}_{k_2}\,|x_2\lambda_2 i_2\rangle. \end{aligned} \tag{19.3}$$

The uncoupled matrix element may be expanded by summing over the complete set of bra–kets

$$\langle x_1\lambda_1 i_1|\,P^{\kappa_1}_{k_1}Q^{\kappa_2}_{k_2}\,|x_2\lambda_2 i_2\rangle = \sum_{x\lambda}\langle x_1\lambda_1 i_1|\,P^{\kappa_1}_{k_1}\,|x\lambda i\rangle\,\langle x\lambda i|\,Q^{\kappa_2}_{k_2}\,|x_2\lambda_2 i_2\rangle. \tag{19.4}$$

Applying the Wigner–Eckart theorem to this, twice, and using various properties of the 3-*jm* symbol, one can combine (19.3) and (19.4) to give

$$\begin{aligned}\langle x_1\lambda_1\|\,\{P^{\kappa_1}Q^{\kappa_2}\}^{r\kappa s}\,\|x_2\lambda_2\rangle = \sum_{x\lambda}&|\kappa|^{\frac{1}{2}}\phi_{\lambda_1} m((12)\,\lambda^*\kappa_2\lambda_2)_{r_2r_2'}m((23)\,\lambda_1^*\kappa_1\lambda)_{r_1r_1'} \\ &\times m((123)\,\kappa_1\kappa_2\kappa^*)_{rr'}\begin{Bmatrix}\kappa_2 & \kappa^* & \kappa_1\\ \lambda_1 & \lambda & \lambda_2\end{Bmatrix}_{r_2'sr_1r'} \\ &\times\langle x_1\lambda_1\|\,P^{\kappa_1 r_1}\,\|x\lambda\rangle\,\langle x\lambda\|\,Q^{\kappa_2 r_2}\,\|x_2\lambda_2\rangle. \end{aligned} \tag{19.5}$$

Note the presence of the two multiplicity indices (r and s) in the reduced matrix element of a coupled tensor. The index r gives the coupling multiplicity ($\kappa_1\times\kappa_2\supset\kappa$), and the index s arises from the Wigner–Eckart theorem ($\kappa\times\lambda_2\supset\lambda_1$).

The various special cases of (19.5) which Judd and Vanagas have found to be useful arise when the operators P and Q have non-trivial actions on certain subspaces only. Let the kets of (19.5) be kets belonging to a product space, constructed from two distinct Hilbert spaces. Let P and Q each act in only one of these two component spaces. For example, let P and Q be orbital and spin operators respectively, and let the kets of (19.5) be the coupled product of orbital and spin kets. The general matrix element can now be written

$$\langle(\lambda_1\lambda_2)\,r_1\lambda i|\,\{P^{\kappa_1}Q^{\kappa_2}\}^{t\kappa}_k\,|(\mu_1\mu_2)\,r_2\mu j\rangle. \tag{19.6}$$

For brevity we have omitted such additional labels as may be required for the complete specification of the kets $|\lambda_1 i_1\rangle$, etc. We assume that the operators act so that

$$P^{\kappa_1}_{k_1}Q^{\kappa_2}_{k_2}\,|\mu_1 j_1, \mu_2 j_2\rangle = (P^{\kappa_1}_{k_1}|\mu_1 j_1\rangle)\,(Q^{\kappa_2}_{k_2}|\mu_2 j_2\rangle). \tag{19.7}$$

These assumptions enable us to evaluate (19.6) in the following fashion

$$\begin{aligned}&(\lambda\kappa\mu)_s{}^i{}_{kj}\langle(\lambda_1\lambda_2)\,r_1\lambda\|\,\{P^{\kappa_1}Q^{\kappa_2}\}^{t\kappa s}\,\|(\mu_1\mu_2)\,r_2\mu\rangle \\ &\quad= |\lambda,\mu,\kappa|^{\frac{1}{2}}\phi_\kappa\phi_\mu(\lambda_1\lambda_2\lambda)_{r_1i_1i_2}{}^i(\kappa_1\kappa_2\kappa)^{tk_1k_2}{}_k(\mu_1\mu_2\mu)^{r_2j_1j_2}{}_j\langle\lambda_1 i_1, \lambda_2 i_2|\,P^{\kappa_1}_{k_1}Q^{\kappa_2}_{k_2}\,|\mu_1 j_1, \mu_2 j_2\rangle. \end{aligned} \tag{19.8}$$

The uncoupled matrix element on the right of this equation simplifies, using (19.7) to the product

$$\langle\lambda_1 i_1|\, P^{\kappa_1}_{k_1}\,|\mu_1 j_1\rangle\,\langle\lambda_2 i_2|\, Q^{\kappa_2}_{k_2}\,|\mu_2 j_2\rangle.$$

A 9-j symbol is given if we use the Wigner–Eckart theorem on this last expression and collect the various 3-jm symbols together. Hence

$$\langle(\lambda_1\lambda_2)\,r_1\lambda\,\|\{P^{\kappa_1}Q^{\kappa_2}\}^{t\kappa s}\|\,(\mu_1\mu_2)\,r_2\mu\rangle = |\lambda,\mu,\kappa|^{\frac{1}{2}}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda^* \\ \kappa_1^* & \kappa_2^* & \kappa \\ \mu_1^* & \mu_2^* & \mu\end{Bmatrix}\begin{matrix} r_1\\ t\\ r_2\end{matrix}$$
$$\begin{matrix} s_1 & s_2 & s\end{matrix}$$
$$\times\langle\lambda_1\|P^{\kappa_1 s_1}\|\mu_1\rangle\,\langle\lambda_2\|Q^{\kappa_2 s_2}\|\mu_2\rangle. \quad (19.9)$$

There are three physically important special cases of this in which the 9-j reduces to a 6-j – either when one of the two operators is not present, or when they are coupled to give an overall scalar. They may be obtained from (19.9) by setting the appropriate representation to zero and using (10.4) or by direct evaluation.

It is worth remarking that if the Hilbert space $\mathscr{H}$ is formed from the tensor product of spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, then a complete basis for the operators of $\mathscr{H}$ can be constructed as the tensor product of the operators of $\mathscr{H}_1$ and $\mathscr{H}_2$.

20. A subgroup basis

In part B we discussed in detail the consequences of choosing the basis for the ket representation by means of a subgroup H. In particular, this led to the Racah factorization lemma and the definition of coupling (isoscalar) and jm factors. We note that when extending these concepts to tensor operators, one must be careful to work with a complete basis set for the Hilbert space.

In part B, the basis kets were labelled by five indices $x\ \lambda\ a\ \mu\ i$ (λ and μ being representation labels of the groups G and H respectively, and a being an index specifying the branching multiplicity). x played the same role for G as before, but the three labels $x\ \lambda\ a$ play this role for H.

A complete set of linear operators for the Hilbert space is (15.1)

$$|x_1\lambda_1 a_1\mu_1 i_1\rangle\,\langle x_2\lambda_2 a_2\mu_2 i_2|. \quad (20.1)$$

Linear combinations are taken as in § 15, to give the standard basis tensors of H

$$(s\mu i(x_1\lambda_1 a_1\mu_1,\, x_2\lambda_2 a_2\mu_2)) = |\mu|^{\frac{1}{2}}\,(\mu_1)^{i_1 i_1'}\begin{pmatrix}\mu_1 & \mu & \mu_2\\ i_1' & i & i_2\end{pmatrix}^{s}|x_1\lambda_1 a_1\mu_1 i_1\rangle\,\langle x_2\lambda_2 a_2\mu_2 i_2|, \quad (20.2)$$

or of G

$$(r\lambda a\mu i(x_1\lambda_1,\, x_2\lambda_2)) = |\lambda|^{\frac{1}{2}}(\lambda_1)^{a_1\mu_1 i_1,\, a_1'\mu_1' i_1'}\begin{pmatrix}\lambda_1^* & \lambda & \lambda_2\\ a_1'\mu_1' & a\mu & a_2\mu_2\\ i_1' & i & i_2\end{pmatrix}^{r}|x_1\lambda_1 a_1\mu_1 i_1\rangle\,\langle x_2\lambda_2 a_2\mu_2 i_2|, \quad (20.3)$$

where it is to be noted that all labels in (20.2) are essential. (We are now using the extended Wigner notation for 3-jm symbols.) We can use the 1-jm and 3-jm factors to write the tensors of (20.3) in terms of those of (20.2)

$$(r\lambda a\mu i(x_1\lambda_1,\, x_2\lambda_2)) = |\lambda|^{\frac{1}{2}}\,|\mu|^{-\frac{1}{2}}\,(\lambda_1)^{a_1\mu_1,\, a_1'\mu_1^*}\begin{pmatrix}\lambda_1 & \lambda & \lambda_2\\ a_1'\mu_1^* & a\mu & a_2\mu_2\end{pmatrix}^{r}_{s}(s\mu i(x_1\lambda_1 a_1\mu_1,\, x_2\lambda_2 a_2\mu_2)). \quad (20.4)$$

The implied sum here is over $a_1'\ a_1\ \mu_1\ a_2\ \mu_2$ and s.

It is important to realize that we are only able to derive (20.4) because the tensors of (20.2) for H, not only already contain information on G, but also span the same space. An example when this is not so, are the operators defined for H by (17.5). These operators are insufficient to enable us to construct such operators of G.

The extension of the Racah factorization lemma to the Wigner–Eckart theorem is straightforward. A tensor defined by (16.2) or (16.3) with respect to G, will also be a tensor with respect to H. The Wigner–Eckart theorem applied to both groups gives,

$$\langle x_1\lambda_1 a_1\mu_1\| Q^{\lambda a\mu s}\| x_2\lambda_2 a_2\mu_2\rangle = (\lambda_1)^{a_1\mu_1,\, a_1'\mu_1^*}\begin{pmatrix}\lambda_1 & \lambda & \lambda_2\\ a_1'\mu_1^* & a\mu & a_2\mu_2\end{pmatrix}^r_s \langle x_1\lambda_1\| Q^{\lambda r}\| x_2\lambda_2\rangle. \quad (20.5)$$

Let us look again at the group operators. Consider (18.5) applied in both G and H, but only for $R\in H$. We cancel the 3-jm symbol for H as well as the representation matrix $\mu(R)$, to give

$$U(R, r\lambda\lambda' a'\mu' i') = \sum_{ab\mu s} |\lambda|^{\frac{1}{2}}\,|\mu|^{-\frac{1}{2}}(\lambda)^{a\mu,\, b\mu^*}\begin{pmatrix}\lambda^* & \lambda' & \lambda\\ b\mu^* & a'\mu' & a\mu\end{pmatrix}^r_s V(R, s\mu\mu' i') \quad \text{for} \quad R\in H, \quad (20.6)$$

where we have indicated all sums explicitly. This equation suggests that if we know how to write some group operators in terms of irreducible tensors, then we will be able to extract the 3-jm factors. Vanagas (1971) uses this technique to great effect for S_n. He studies the group-algebraic properties of the interchange operators $(r, r-1)$, the matrix elements of which are known by Yamanouchi's construction (Hamermesh 1962, p. 214). He produces sufficient equations to solve for all the 3-jm factors required in his nuclear calculations.

21. Conclusion

In this article we have carefully distinguished between properties derived for the coupling coefficients and tensor operators of all (finite or compact) groups, properties true for special groups, and properties that may be obtained by applying requests of simplicity. The matrices U of (3.2) and K of (5.1) are fundamental in this regard. The matrix K is the generalization to a non-multiplicity free group of the familiar arbitrary phase of the multiplicity free case. To emphasize this it is appropriate to refer to it as the 'phase matrix'.

As this article has developed, various flaws have become apparent in the various notations that are often widely used. In particular, we would strongly recommend the adoption of the name 3-jm symbol for what is usually called a 3-j symbol. Indeed to be fully consistent with the terminology, the 1-j phase ϕ_λ should be denoted (for it relates λ and λ^*)

$$\phi_\lambda = \{\lambda\} \quad 2\text{-}j \text{ symbol},$$

and the 3-j permutation matrices

$$m(\pi, \lambda_1\lambda_2\lambda_3)_{rs} = \{\pi, \lambda_1\lambda_2\lambda_3\}_{rs} \quad 3\text{-}j \text{ symbol}.$$

Thus a 9-j symbol with one representation the scalar, reduces to a 6-j symbol (10.4), a 6-j to a 3-j (9.17) and a 3-j to a 2-j (5.10). All the n-j symbols are independent of the bases chosen for the representations and are written with curly brackets.

The jm symbols are quite different. They are dependent on the basis, factorizing for a group theoretic basis. Round brackets are used to display this difference. It would be sensible to rename the 1-*jm* symbol a 2-*jm* symbol. This is because the symbol depends on two representations and their bases, we would then use the notation

$$(\lambda)_{i_1 i_2} = \begin{pmatrix} \lambda & \lambda^* \\ i_1 & i_2 \end{pmatrix} \quad \text{2-}jm \text{ symbol.}$$

We then have that the 2-*j* symbol gives the reordering property of the 2-*jm* symbol, the 3-*j* symbol for the 3-*jm*, 6-*j* for the 4-*jm*, 9-*j* for the 5-*jm*, etc. For example, one can see that the coupling coefficient of (9.4) relates different coupling schemes between the four representations $\lambda_1 \lambda_2 \lambda_3$ and λ. This viewpoint is very nicely expressed by figure 22 of Agrawala & Belinfante (1968).

Various results in this article suggest that all *j* symbols and *jm* factors can be chosen real. The restrictions imposed on *j* symbols by choosing the multiplicity metric, A, unity, and the restrictions of a subgroup on the *jm* factors suggest that real coefficients may always be found. The author has, however, been unable to prove this is always the case, or find evidence that a particular coefficient is necessarily complex.† Harnung (1973), in his discussion on the icosahedral group, remarks that some workers claim complex coefficients are necessary, when they are not, and others assume reality without proof. Consider the following example.

The product [100] × [110] of SO_6 is the product of two orthogonal representations, and contains four irreducible components, the orthogonal representations [100] and [210], and the complex conjugate pair [111] and [11-1]. If one chooses bases of [100] and [110] to give real representation matrices (for this is possible) then it is clearly impossible, using real coupling coefficients in (3.6), to project out complex representation matrices. However, we noted in the example of § 4, that it is not usual to choose real representation matrices anyway.

Further evidence for this reality hypothesis comes from the work of Biedenharn *et al.* (see appendix) on U_3 and SU_3. Their work produces real coefficients for all products in these groups. Both orthogonal and complex representations occur here (although one needs to go to SU_6 to obtain all three kinds of representations).

We would strongly urge that all calculations are carried out by using the 'sensible' choice of (5.2) for the coupling (Wigner) coefficients and the coupling (isoscalar) factors, and not the Condon & Shortley (1935) or Racah (1949) phases.

There are several areas of recent interest that have not been mentioned in this article. First, it is clear from (13.12) that the 3-*jm* factors will be zero if the labels $\lambda_1 a_1 \mu_1$ and $\lambda_2 a_2 \mu_2$ are the same, but the appropriate interchange matrices (for G and H) are different. Judd (1971) has studied the zeros occurring in the various 3-*jm* factors of interest to atomic physicists. He is able to explain most zeros by this means.

We have discussed the symmetries that are a consequence of complex conjugation, and of permutations in the order of the product. The adjoint representations of the symmetric groups – those pairs of representations obtained by interchanging the rows and columns of the Young tableaux – are also simply related. Hamermesh (1962, p. 265) shows that the appropriate 3-*jm* symbols are similar. Although adjoint representations of other groups are not so simply related, it may be that there is a corresponding symmetry in the *j* symbols. Again, Robinson (1970, 1972) suggests that the famous Regge symmetries for SU_2 (Regge 1958, 1959; Kramer & Seligman 1969) have a counterpart for all groups.

In part C we have extended the ideas developed in the first two parts concerning the action of a group on a Hilbert space, to the action of the group on the space of operators on the Hilbert

† *Note added in proof, 7 October* 1974: complex coefficients do occur, beginning with the tetrahedral group.

space. In doing this it was especially important to use a complete basis for the operator space. After writing down such a basis as $|x_1\lambda_1 i_1\rangle\langle x_2\lambda_2 i_2|$ and transforming to the irreducible basis $(r\lambda i(x_1\lambda_1, x_2\lambda_2))$, the Wigner–Eckart theorem and the other results followed rather easily.

The group operators, O_R, being a subset of the set of all linear operators on our Hilbert space, were easily written in terms of the irreducible tensor operators. This step was very important in relating the various coefficients to the representation matrices in a way which leads to recursive formulae for the coefficients for a chain of groups. This idea has been fruitfully pursued by Jucys, Vanagas and others (see Vanagas 1971; Alisauskas, Jucys & Jucys 1972; for detailed references).

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Some of the points in the appendix I owe to Dr L. C. Biedenharn, while he was a Visiting Erskine Fellow at the University of Canterbury.

Appendix: The Biedenharn–Louck tensors for U_N

In a lengthy series of articles Biedenharn and his co-workers have studied the properties of the coupling and recoupling coefficients of the unitary groups (Biedenharn 1963; Baird & Biedenharn 1963–5). Recently several survey articles have also appeared (see, for example, Louck 1970; Holman & Biedenharn 1971) and also substantial applications of theory to U_3 (Biedenharn, Louck, Chacon & Ciftan 1972; Biedenharn & Louck 1972). The purpose of this appendix is to use the present language to discuss this formulation and a few of the results they have obtained.

As these authors note, the group U_N has some rather special properties, and I believe it useful to carefully distinguish the general from the specific.

First, U_N is a direct product group $U_N = SU_N \times U_1$, and thus it follows that the 3-*jm* symbols (and coupling coefficients) of U_N may be written as a product of 3-*jm* symbols for the two groups SU_N and U_1. The 3-*jm* symbol for U_1 is especially simple. Labelling representations of U_1 (all one-dimensional) in the usual fashion by a single integer n (= $2m$ of SO_2) we have for all 3-*jm* symbols

$$\begin{pmatrix} n_1 & n_2 & n_3 \\ 1 & 1 & 1 \end{pmatrix} = \delta_{n_1+n_2+n_3,\,0}, \tag{A 1}$$

or this multiplied by a completely arbitrary phase. (A 1) is much the simplest choice and the present author would take the view that it should be chosen always. 'Always' is most important since we have then related the phases of all 3-*jm* symbols of U_N to those of SU_N, where there is much less choice (see also our comments after (14.3), and in Kramer 1968, section 7).

Secondly, Louck & Biedenharn use the canonical subgroup basis $(U_n \supset U_{N-1} \supset \ldots \supset U_1)$ throughout. We have shown here that certain properties of the coefficients are a consequence of the group, but other properties follow from the particular basis chosen. In particular the recoupling coefficients are independent of the basis choice (see our remark after equation (11.5)). Thus if a canonical labelling scheme for the multiplicity space exists for a particular group one might hope to derive this fact without discussing the ket basis. We are suggesting that one should aim to calculate the recoupling coefficients before the coupling coefficients.

Thirdly, partly as a consequence of their basis choice, Louck and Biedenharn attach considerable significance to the following result. Writing the resolution of the Kronecker product of U_N as

$$\{\lambda\}\times\{\mu\} = \Sigma_\nu g_{\lambda\mu\nu}\{\nu\} \tag{A 2}$$

we have

$$\Sigma_\nu g_{\lambda\mu\nu} \leqslant |\{\lambda\}_{U_N}| \tag{A 3}$$

(the right-hand side being the dimension of the representation $\{\lambda\}$). The equality holds for some $\{\mu\}$. These results suggested to Biedenharn that one should label the terms in the product by the labels used for the basis kets of the ket-space $\{\lambda\}$ (the Gel'fand pattern). However, we have also that the Littlewood–Richardson rule gives

$$g_{\lambda\mu\nu} \leqslant |[\lambda]_{S_l}|, \tag{A 4}$$

where the equality holds for some μ, ν and N. (A 4) would suggest labelling the multiplicity using a Yamanouchi label – that is, a basis label appropriate to the symmetric group S_l. Other considerations support this latter suggestion. First, the form of (A 3) suggests a similar result for other groups, which is not the case, but (A 4) depends on the well known duality of the properties of the unitary and symmetric groups. This duality is peculiar to these groups and would not be expected to generalize.

Further support for such a multiplicity label comes from the work of Kramer (1968). Although considering only the multiplicity free case, he proves that the 6-j symbol for U_N is equal (up to phases and normalizations) to a 6-f symbol for S_l. The 6-f symbol is defined by recoupling between subgroup chains. A Yamanouchi type symbol may thus arise in a very natural fashion in the general case.

None of these remarks should be taken as evidence that the two labelling schemes do not in fact coincide. We repeat our remark that the unitary and symmetric groups have all sorts of combinatoric structure in addition to what may be regarded as group theoretic in origin. (One of Biedenharn and Louck's earlier problems was to assign the operator patterns to their null spaces. The null space concept separates the multiplicity space in a clear-cut manner, but only fixes the operator pattern up to its 'shift' value. Note that the number of non-null patterns Γ of λ with given shift, μ–ν, is precisely $g_{\lambda\mu\nu}$. In the U_3 case and certain other cases they prove that the limit properties of the coefficients do provide a unique assignment of the operator pattern.)

In future papers we hope to explore the duality of the symmetric and the unitary groups. The interwovenness of the theories of these groups can perhaps be understood as resulting from the unitary group essentially describing the most general linear transformation, and the symmetric group describing the most general permutation. In this connexion it is apparent also that further combinatorial results can be deduced from the character theory of these groups (see Butler & King 1973 *a*, *b*).

Let us now go on to discuss the vector space concepts used by the Biedenharn–Louck school, for they manipulate the spaces in a more powerful manner than is usual, and do not always state this. In particular they usually identify (in the manner of a pure mathematician) all spaces which have the property under review. For example, in Louck (1970, p. 20) Louck chooses the linear space the tensors operate on to contain each and every type of irreducible representation space (an infinite number) once and only once. In a physical situation this would be realized but rarely and we usually need to retain, for use elsewhere, any non-group-theoretic properties of the spaces. In this article, we have explicitly included the label x to distinguish similar representation spaces. When restriction is made to a subgroup (in order to obtain recursion formulae) a corresponding restriction is made by these other authors, from the Hilbert space to a subspace. The subspace is again required to contain each dissimilar ket representation space exactly once. The manner of choosing (by projection or factorization) such a subspace from the infinitely many which are available, is most instructive, so let us discuss this now.

In order to see the generality of their construction and to ignore properties consequent upon their canonical basis, let us return to the general situations of §§ 2 and 12. We take a Hilbert space $\mathscr{H}$ with a group G acting upon it. From basic group representation theory (applicable to any finite or compact group) we may write $\mathscr{H}$ as a direct sum of (finite dimensional) irreducible representation spaces.

$$\mathscr{H} = \bigoplus_{x\lambda} \mathscr{H}_{x\lambda}, \tag{A 5}$$

where x is a discrete label and is independent of the group properties, but 'counts' the irrep spaces labelled λ. Let there be k_λ of them. Thus the space

$$\bigoplus_{x=1}^{k_\lambda} \mathscr{H}_{x\lambda}, \tag{A 6}$$

with basis

$$\{|x\lambda i\rangle : x = 1, \ldots k_\lambda,\ i = 1, \ldots |\lambda|\}, \tag{A 7}$$

is factorizable, into a tensor product space

$$\bigoplus_{x=1}^{k_\lambda} \mathscr{H}_{x\lambda} = U_\lambda \otimes \mathscr{H}_\lambda, \tag{A 8}$$

with bases

$$\{|x\lambda\rangle : x = 1, \ldots, k_\lambda\} \quad \text{for} \quad U_\lambda, \tag{A 9}$$

and

$$\{|\lambda i\rangle : i = 1, \ldots |\lambda|\} \quad \text{for} \quad \mathscr{H}_\lambda, \tag{A 10}$$

where

$$|x\lambda i\rangle = |x\lambda\rangle\, |\lambda i\rangle, \tag{A 11}$$

and the $\mathscr{H}_\lambda$ contains all the group theoretic information (of G).

For a subgroup H of G, the space $\mathscr{H}_\lambda$ reduces to a direct sum of irreps of H

$$\mathscr{H}_\lambda = \bigoplus_{a\mu} \mathscr{H}_{\lambda a\mu}, \tag{A 12}$$

where the range of the sum over a is from 1 to α^λ_μ, the multiplicity of μ in λ. By the above argument (H taking the place of G, and a of x) we have

$$\mathscr{H}_\lambda = V^\lambda_\mu \otimes \mathscr{H}_\mu, \tag{A 13}$$

the bases being

$$\{|\lambda a\mu\rangle : a = 1, \ldots, \alpha^\lambda_\mu\} \quad \text{for} \quad V^\lambda_\mu, \tag{A 14}$$

and

$$\{|\mu j\rangle : j = 1, \ldots |\mu|\} \quad \text{for} \quad \mathscr{H}_\mu. \tag{A 15}$$

A typical basis ket of the original Hilbert space $\mathscr{H}$ may then be written

$$|x\lambda a\mu j\rangle = |x\lambda\rangle\, |\lambda a\mu\rangle\, |\mu j\rangle, \tag{A 16}$$

where we are to remember that the kets belong to four different spaces. From our work in part B of this paper, it is clear that the spaces V^λ_μ, with typical basis kets $|\lambda a\mu\rangle$ are the most interesting, even though in the multiplicity one case they are one-dimensional.

One of the important aspects of the Biedenharn–Louck approach is to factor the tensor operators of the basis (15.5) into novel pieces. The essential point is to take an incomplete operator basis for a Hilbert space constructed in such a way from the factored pieces above (A 16), that the operators are a basis for our original space.

As before, we first ignore the subgroup structure. (A 5) and (A 8) give us

$$\mathscr{H} = \bigoplus_{\lambda} (U_\lambda \otimes \mathscr{H}_\lambda), \tag{A 17}$$

and this is a subspace of

$$\mathscr{H}' = U_G \otimes \mathscr{H}_G, \tag{A 18}$$

where

$$U_G = \bigoplus_{\lambda} U_\lambda, \tag{A 19}$$

and

$$\mathscr{H}_G = \bigoplus_{\lambda} \mathscr{H}_\lambda. \tag{A 20}$$

Now a basis for the operator space on U_G is the set of

$$|x_1\lambda_1\rangle\langle x_2\lambda_2|. \tag{A 21}$$

From the set of all basis operators of $\mathscr{H}_G$

$$(r\lambda i(\lambda_1, \lambda_2)) \tag{A 22}$$

we select a subset by taking certain linear combinations. Either the S-functional or weight diagram approaches to the character theory of the classical groups leads to natural significance of the difference

$$\Delta \equiv \lambda_1 - \lambda_2$$

of the representation labels (but here we only require countability). Biedenharn and Louck combine this 'shift' Δ with the index r, to produce an 'operator pattern' $\Gamma \equiv r\Delta$ which for the unitary groups may be usefully put into correspondence with the canonical basis label (but again this correspondence is not relevant to this present discussion on factorization).

The canonical unit (Wigner) tensor operators of Biedenharn and Louck are obtained from (A 22) by summing over λ_2 for fixed Γ

$$\left\langle \begin{matrix} \Gamma \\ \lambda \\ i \end{matrix} \right\rangle = \sum_{\lambda_2} (r\lambda i(\lambda_1, \lambda_2))$$

$$= \sum_{\lambda_2} \sum_{i_1, i_2} \langle r\lambda_1 i_1 | \lambda i; \lambda_2 i_2\rangle |\lambda_1 i_1\rangle\langle\lambda_2 i_2|. \tag{A 23}$$

The tensor products of the operators (A 21) and (A 22) are clearly a basis for the operator space on (A 18). The tensor products of the operators (A 21) and (A 23) generate a subspace of this, but are a basis of the operator spaces of (A 17), the original Hilbert space. We have

$$(r\lambda i(x_1\lambda_1, x_2\lambda_2)) = |x_1\lambda_1\rangle\langle x_2\lambda_2| \left\langle \begin{matrix} \Gamma \\ \lambda \\ i \end{matrix} \right\rangle. \tag{A 24}$$

(We use (A 11) to prove the matrix elements are identical.)

We now return to the subgroup structure, $G \supset H$. A repeat of the argument gives

$$\left\langle \begin{matrix} \Gamma \\ \lambda \\ a\mu \\ j \end{matrix} \right\rangle = \sum_{\gamma} \begin{bmatrix} \Gamma \\ \lambda \\ a\mu \\ \gamma \end{bmatrix} \left\langle \begin{matrix} \gamma \\ \mu \\ j \end{matrix} \right\rangle, \tag{A 25}$$

where for the subgroup we have

$$\gamma = s\delta, \quad \delta = \mu_1 - \mu_2$$

and

$$\begin{pmatrix} \gamma \\ \mu \\ j \end{pmatrix} = \sum_{\mu_2} (s\mu j(\mu_1, \mu_2)). \tag{A 26}$$

The tensor operator factor

$$\begin{bmatrix} \Gamma \\ \lambda \\ a\mu \\ \gamma \end{bmatrix}$$

is defined by using a coupling (or 3-*jm*) factor

$$\begin{bmatrix} \Gamma \\ \lambda \\ a\mu \\ \gamma \end{bmatrix} = \sum_{\lambda_2} \sum_{a_1 a_2 \mu_2} \langle r\lambda_1 a_1 \mu_1 s | \lambda a\mu;\ \lambda_2 a_2 \mu_2 \rangle\, |\lambda_1 a_1 \mu_1\rangle \langle \lambda_2 a_2 \mu_2|. \tag{A 27}$$

Biedenharn & Louck term the operator factor a 'projective unit Wigner tensor operator'. For any canonical basis the labels a are not needed, and Biedenharn & Louck suppress the μ (by including it in γ) for the operator factor. We prefer not to do this.

The Biedenharn–Louck operator factor has an important uniqueness property for a canonical (multiplicity zero or one) basis. Namely the action of an operator on a ket is to produce a single basis ket, with a numerical factor, which is the coupling factor. An otherwise more natural operator factor might seem to be obtained by inserting an independent sum over μ_1 in (A 27) – that is, by fixing s and not $\gamma = s\delta$. Such an operator factor would be more simply related to the basis tensors of the space $\oplus V^{\lambda}_{\mu}$. We would have

$$\begin{bmatrix} \Gamma = r\Delta \\ \lambda \\ a\mu \\ s \end{bmatrix}' = \sum_{\lambda_2} (r\lambda a\mu s(\lambda_1 \lambda_2)), \tag{A 28}$$

and the sum over $\gamma = s\delta$ in (A 25) would become one over s only. Symmetry can be preserved by similarly inserting sums over λ_1 in (A 23) and (A 28), and μ_1 in (A 26).

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